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Computing risk measures for non-normal asset returns using Copula theory

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ABSTRACT

The article investigates the long memory effect on risk measures such as Value at Risk (VaR) and Conditional Value at Risk (CVaR). In addition to a more realistic representation of data, our results affirm that much more reliable conclusions will certainly be drown if a more classes of Copula functions can be used.

Keywords: Value-at-Risk, Conditional Value-at-Risk, Extreme Value Theory, Long memory, Copula.

JEL Classification: C15, C22, G15.

1. Introduction

Portfolio optimization is one of the most attracting fields in decision-making. The benefit of using the variance in the mean-variance Markowitz's formulation, for quantifying portfolio risk, is principally due to the simplicity of the computation. Many researchers such as Bai et al. (2007) affirm that in modern financial analysis, evidence of non normality of the distribution of financial return variables grows every year. As far as the normality assumption is concerned, it is also often made for statistical inference.[1] Value at Risk (VaR) is one of the most popular measures due to its simplicity, which has achieved the high status of being written into industry regulations. However, as a risk measure, VaR has recognized limitations. Firstly it lacks subadditivity and convexity (Artzner et al. (1999)). Indeed, VaR is a coherent risk measure only when it is based on the standard deviation of elliptical distributions. In addition, it has been shown in Andersson et al. (1999) that the problem of minimizing VaR of a portfolio can have multiple local minimizes. Artzner et al. (1999) propose the main properties that a risk measures must satisfy, thus establishing the notion of coherent risk measure. Conditional Value-at-Risk, or CVaR for short, is defined as the weighted average of VaR and losses strictly exceeding VaR for general distribution (Rockafellar et al. (2002)). The CVaR risk measure has been proved to be a coherent risk measure in many studies such as Pflug et al. (2000), Acerbi et al. (2001). After that, other classes of measures have been proposed, each with distinctive properties: Conditional Drawdown-at-risk (CDaR) in Chekhlov et al. (2000), ES in Acerbi et al. (2001), convex measures in Follmer et al. (2002), and deviation measures in Rockafellar et al. (2006).

Clearly, the VaR and the CVaR of a portfolio depends on the behavior of the individual assets in the portfolio and also on the dependence structure between them. In particular, the dependence in the tails of the distribution strongly influences the VaR and CVaR calculation (Embrechts et al. (1999) and Kiesel et al. (2002)). Thus, the correlation coefficient, which is not adequate to measure the dependence in the tails, may lead to inaccurate estimations of VaR and CVaR. Alternatively, Ausin et al. (2010) affirm that copulas provide a useful tool to model tail dependence and obtain precise VaR and CVaR estimations. Besides the applications of VaR and CVaR in risk measurement, they also provide useful tools in optimal portfolio selection. Several research papers used various types of Copulas, VaR and extreme value theory to study behaviors of financial variables such as Li et al (2012) and Chaithep et al (2012).

As introduced in Agrawal (2008), copula theory effectively captures the non-linear inter-dependence. Based on Student t-copula, assuming marginal distribution as Gaussian in the center and EVT distribution in the tail, the author compute the market risk using the VaR and CVaR measures. This techniques neglect the long range dependence, always detected in financial data. Our aim is to prove that it a novel

[1] Gourieroux, 2005

method but it model both market prices and shocks persistence in their, which distort investors' preferences.

This paper is organized as follows. In section2 and section3, we expose brief definitions of respectively Copulas theory and long memory process. Next, we present our empirical results. The final section is the conclusion.

2. Copula for Modeling Dependence

Our basic reference in this section is the famous book of Nelsen (1999), second edition, titled "An introduction to copulas". We begin with the definition of copulas and formulation of theorem of Sklar mentioned in the introduction (Nelsen (1999), Embrechts et al. (2002), McNeil et al. (2005), Nelsen (2006)). Theorem of Sklar elucidates the role that copulas play in the relationship between multivariate distribution functions and their univariate margins.

Theorem of Sklar (1959): Let F is a joint distribution function with bivariate marginal distributions F_1 and F_2. Then there exists a 2-dimensional Copula C such that for all $(x_1,x_2) \in [-\infty, +\infty]^2$

$$F(x_1, x_2) = C(F(x_1), F(x_2))$$

Furthermore, if the marginals are all continuous, C is unique. Conversely, if C is a copula and F_1 and F_2 are distribution functions, then the function \$F\$, initially mentioned, is a joint distribution function with margins F_1 and F_2.

This theorem first appeared in Sklar (1959). The name "copula" was chosen to emphasize the manner in which a copula "couples" a joint distribution function to its univariate margins.[2]

Statistically, a two-dimensional Copula is a function C that has the following properties:

1. $DomC = [0,1] \times [0,1]$
2. $C(0,u) = C(u,0) = 0$ and $C(1,u) = C(u,1) = u$, $\forall u \in [0,1]$
3. C est 2-increasing:

   $C(v_1,v_2) - C(v_1,u_2) - C(u_1,v_2) + C(u_1,u_2) \geq 0$, $\forall (u_1,u_2) \in [0,1]^2$ and when $0 \leq u_1 \leq v_1 \leq 1$ and $0 \leq u_2 \leq v_2 \leq 1$.

Let $U = (u_1,u_2)$ be random vector with u_1 and u_2 are uniform random variables, then we have:

$$C(u_1,u_2) = \Pr(U_1 \leq u_1, U_2 \leq u_2)$$

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[2] See Nelsen (2006).

Copulas allow us to consider more general multidimensional distributions independently of marginal distributions and hence the implementation of multivariate models becomes easier.

3. Long Memory Process

The long memory, introduced in the early 1950s, has changed the degree of market development efficiency with Hurst work. Further developments have been made by Mandelbrot(1969a), Mandelbrot et al. (1972), Granger (1980) and later on by Hosking (1981). The study of this property is generally based on the ARFIMA model (when long memory is in mean) and on the Fractionally Integrated GARCH (FIGARCH) process (when long memory is in the volatility process).

3.1. ARFIMA process

A stochastic process with stationary and second order $(X_t)_t$ is defined as a long memory if ρ decreases hyperbolically to zero, where ρ is the autocorrelation function. Formally, if there is a reel α as 0< α <1 and constant c, with c> 0, verifying:

$$\lim_{x \to \infty} \frac{\rho_h}{c.h^{-\alpha}} = 1$$

Long memory can easily be defined by considering the spectral density of the process $(X_t)_t$ with frequency λ, noted f (λ). A stochastic process with stationary and second order $(X_t)_t$ is long memory if there is a frequency λ_0 that its spectral density is not bounded, namely: $f_X(\lambda) \approx C_f(\lambda).(\lambda - \lambda_0)^{-2d}$, $\lambda \to \lambda_0$ where $C_f(.)$ designates a function that varies slowly on λ_0 and 0<d<1.

If $(X_t)_t$ is an ARFIMA (p, d, q) which may be presented as follows:

$$\Phi(L) X_t = \Theta(L) \varepsilon_t$$

With $\varepsilon_t = (1-L)^{-d} \eta_t$, η_t is a white noise with variance σ^2 and $\Phi(L)$ and $\Theta(L)$ are polynomials of degree respectively p and q.

3.2. FIGARCH process

It is introduced by Baillie (1996). Formally, a FIGARCH (p, q) process is given by:

$$(1-L)^d \alpha(L) \sigma_t^2 = \omega + \beta(L) \eta_t$$

η_t : denotes a white noise.

d: designates parameter of fractional integration.

$\alpha(L) = \alpha_1 L^1 + ... + \alpha_p L^p$ and $\beta(L) = \beta_1 L^1 + ... + \beta_p L^p$ being two polynomials with roots greater than unity.

The FIGARCH model offers a direct measure of persistence through fractional integration parameters. Moreover, the effect of a volatility shock is reduced to a hyperbolic component in time, which allows better identifying long-term volatility.

The fractional difference parameter d allows the indirect measurement of long-term persistence of volatility.

4. Empirical results

The empirical investigation uses monthly financial data such as the Standard and Poor's 500 COMPOSITE, Dowjones Industrial (DJIND) and CAC40 French index, for the period going from 01/01/1999 to 25/03/2011. It is necessary to have a stationary series. Thus we are interested in logarithm daily prices, resulting in a total of 3190 observations.

TABLE1: Statistics of the data

	DLCAC40	DLDJIND	DLSP500
Mean	1,1313E-004	8,8350E-005	8,8027E-005
variance	2,3083E-004	1,5402E-004	1,7637E-004
Maximum	0,1059	0,1051	0,1096
Minimum	-0,0947	-0,0820	-0,0946
Skewness	0,0534	-0,0039	-0,1123
Kurtosis	8,1910	10,7880	10,7105

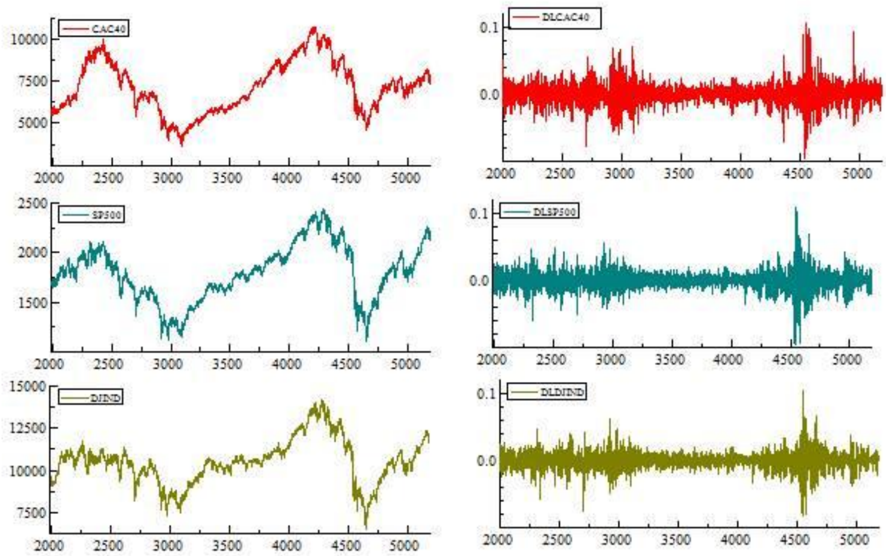

FIGURE1: The row data and their first difference of log

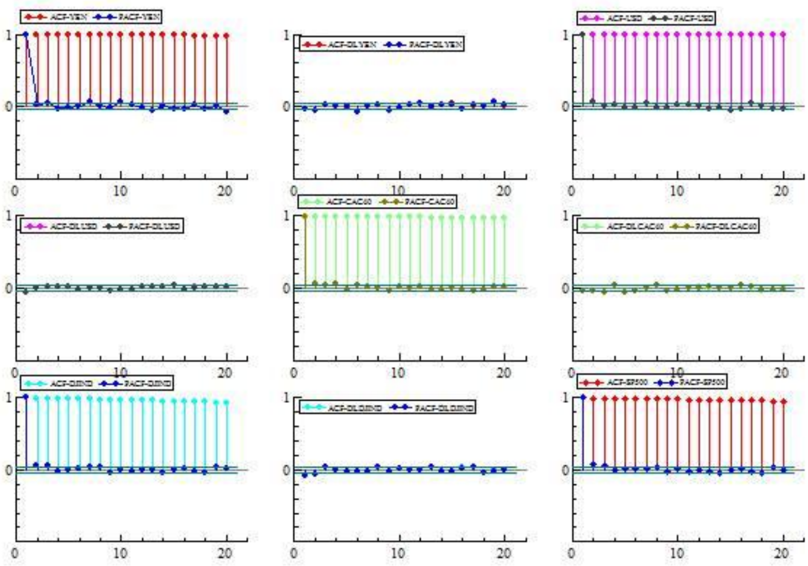

FIGURE2: The ACF and PCF of the series

4.1. Modeling tails of asset distributions

The tail modeling using an Extreme Value distribution requires the data to be independent and identically distributed (i.i.d.). When assessing interdependencies, it is important to take into consideration the property of asymmetric propagation of shocks. For a given pair of financial returns, their joint (row) excesses typically are not identically distributed (i.d.). As several studies, we may interpret the observed asymmetry in the markets joint behavior as an intrinsic characteristic not just due to volatility. We provide examples of such situations. The next table summarizes the GPD's parameters of each financial distribution. The parametric estimation of GPD is done using the Maximum Likelihood technique.[3]

TABLE2: Estimated parameters of tail distributions

	Xi	Beta	funval
DLCAC40	-0.0393	0.0132	-336.3414
DLDJIND	0.1942	0.0093	-348.3189
DLSP500	0.2164	0.0097	-341.8847

4.2. Long memory estimations

Several studies highlight the property of long memory in the variance process. It is usually characterized by the FIGARCH model.[4] Though, Baillie et al. (1996) affirm that it must verify some conditions such as the positivity of the conditional variance and the

[3] We estimate all the tail distributions using 100 observations.

[4] Before assuming long memory, we apply the test of Wald about the existence of an IGARCH and different long memory test such as R/S test, variance-ratio test.

stationarity of the process. When estimating, the positivity constraint for the FIGARCH model is observed. We don't have regressors in the conditional mean. The variance equations include the FIGARCH (0, d, 0) model which is estimated with BBM's method (Truncation order: 1000). We don't have regressors in the conditional variance. The residual distribution is supposed to be a normal distribution. The results are regrouped in next table.[5]

TABLE3: Estimation results

	DLCAC40	DLDJIND	DLSP500
Cst(M)	0.00069 (0.00018)	0.000393 (0.00015)	0.0004 (0.000135)
AR(1)	0.65133 (0.10034)	-0.042337 (0.0174)	0.71235 (0.09425)
MA(1)	-0.7089 (0.10639)	-	-0.7711 (0.08719)
Cst(V)	-	0.014729 (0.00522)	0.01657 (0.00601)
d-FIGARCH	0.44957 (0.045516)	0.849486 (0.04093)	0.833175 (0.0426)
GARCH(Beta1)	0.419055 (0.05382)	0.844635 (0.02655)	0.83753 (0.03235)

4.3. Remarks and Discussion

We estimate the marginal cumulative distribution function (cdf) for each equity index by fitting a nonparametric Gaussian kernel in the interior of the data and a parametric GPD (Generalised Pareto Distribution) in the upper and lower tails of the data (thresholds are evaluated so that to reserve 10% of data for each tail). For fitting a copula, we firstly transform the marginals into Uniform [0,1] and then fit a copula to the transformed data. We attempt to use two different ways of fitting copula. First, we compute the student copula as {Patton (2006a) and put the estimated coefficient in the risk measure. Second, we proceed as Agrawal (2008) and assess the correlation coefficient empirically using the estimation of the Kendall's tau; estimate the copula associated with the data-series, simulate N times from this copula (where N is large), form N equally weighted portfolios, compute the risk measure for the simulated returns for all N portfolios, compute N empirical estimates of the risk measure and compare it with the above data to check the validity. The detailed statistics of the comparison can be found in the next table.

TABLE4: Copula's Estimation using Elliptical Copula

Equally portfolios	dlsp-dlcac	dlcac-dldjind	dlsp-dldjind
Gaussian copula	0.511385906	0.499402331	0.8926
Student copula	[0.5139;2.6462	[0.4985;2.6018]	[0.8875;3.2691]
λ_L	0.3455	0.3402	0.6389
Kendall’s tau	0.3357	0.3255	0.6926
ρ	0.5044	0.4905	0.887

[5] Values in parenthesis are the standard error for the estimated parameters.

The Gaussian copula is, in effect, a student copula with infinite degrees of freedom. COPULARND generates values on the unit hypercube, whose marginal distributions are uniform. Our estimated degrees are lower; that's why the student is more adequate than the normal. But, we do not forget that even if student is more appreciate than the gaussian Copula; it uses the same classical correlation coefficient as a copula parameter and it is a part of the elliptical class of copula functions. Thus, we are always restrictive when using elliptical copula to model the dependence structure. Our aim is to turn to the archimedean copulas which can take into consideration tail distributions[6] and generalize the concept of structure dependence. We attempt to estimate the dependence structure using different static copula functions such as Clayton, HRT, Gumbel, Rotated Gumbel in addition to Student Copula.[7] The results are summarized in next table.

TABLE5: Copula's estimation

	DLSP500-DLCAC40	DLSP500-DLDJIND	DLCAC40-DLDJIND
Optimal copula	Frank	Frank	Frank
Copula parameter	181.17	448.35	174
λ_U	0	0	0
λ_L	0	0	0

This does not occur an asymptotic dependence in tail distributions. According to Kharoubi-Rakotomalala (2008), this dependence structure gives more weight to the center of the distribution than the Gaussian copula. Thus, with Frank copula, centralized returns are more correlated than extreme events. Diversification is more important than one Gaussian copula. Our result suggests that fat-tailed data does not necessarily have dependence structure which take into consideration dependence of extreme events. This result is maintained even if we introduce dynamic copula such as time-varying normal Copula, time-varying rotated Gumbel copula and time-varying SJC copula.

Finally, we can compute VaR and CVaR on the conditional distribution on vectors from the selected copula[8]. Because of copula functions are non-bijective, we will use the algorithm of Genest et al. (1986) to compute conditional distributions.[9]

[6] Left tail is more consistent than the right one, so dependence structure can not be a symmetric copula function.
[7] HRT and Gumbel have a very closed characteristics and the same for Clayton and Rotated Gumbel.
[8] We selected the optimal copula using AIC, BIC and Log Likelihood criteria.
[9] From the selected copula, we fixe arbitrary one of the two transformed series V and calculate the conditional distribution U when V=v.

TABLE6: Risk measures using elliptical copulas

Equally portfolios	dlcac-dlsp	dlcac-dldjind	dlsp-dldjind
Gaussian VaR1%	-0.207575	-0.241097	-0.239411
Gaussian VaR5%	0.092857	0.089128	0.098605
Gaussian CVaR1%	-0.49360	-0.49163	-0.523948
Gaussian CVaR5%	-0.114684	-0.115124	-0.107483
Student VaR1%	-0.041235	-0.040116	-0.037819
Student VaR5%	-0.022839	-0.022298	-0.020549
Student CVaR1%	-0.055482	-0.05353	-0.052369
Student CVaR5%	-0.034430	-0.033350	-0.031085

The results of Agrawal show that for an equally weighted portfolio, the VaR estimates provided by the t-copula are much closer to the empirical VaR value for 95% confidence levels. Also, the author affirms that gaussian copula consistently under-estimates both the risk measures VaR and CVaR at the mentioned confidence level. The t-copula however, produces results in close matching, its just that the variability of results is higher than that of its Gaussian counterpart; due to the fat-tailed nature of marginals. However, the research of Kapil Agrawal in 2008 is significantly restrictive. Indeed, the author concentrates on an elliptic copula which is the Student copula. Statistically, our goal is to identify the dependency structure of "real" between variables but do not require, even if the marginal follow the Student. Our study proves that if we leave detect the dependency structure using different classes (elliptical Archimedean), the latter may not necessarily be the Student copula. In addition, it is true that our goal is better modeling financial time series, but we must not forget that the observed series are noisy by the effects of crises. This may confuse investors by over or under estimation of risk. Our application shows that the two risk measures based on the filtered series are widely different from those based on noisy series.

TABLE7: Risk measures using row and filtered data

Equally portfolios	VaR 1%	VaR 5%	CVaR 1%	CVaR 5%
dlcac-dlsp	-0.037991765	-0.020051602	-0.048410127	-0.030445897
dlcac-dldjind	-0.037258693	-0.019071237	-0.04673039	-0.029372758
dlsp-dldjind	-0.036292113	-0.020073506	-0.052078619	-0.030807756
filtered dlcac-dlsp	-2.666617805	-1.826327832	-3.340567168	-2.39137266
filtered dlcac-dldjind	-2.686825184	-1.815973644	-3.36422503	-2.382704936
filtered dlsp-dldjind	-2.633205231	-1.777082316	-3.34498234	-2.354218377

Also, many remarks can be addressed to the work of Agrawal (2008). Indeed, the author considers the student copulas as a non-elliptical distribution.

However, literature focusing on copulas affirms that Gaussian copulas and student copulas are the only two types of copulas in the elliptical class[10]. Moreover, on the one hand, Kapil Agrawal affirms that we must close to more realistic representation of data. On the other hand, the author filter the data using an AR (compensates for autocorrelation) and GARCH (compensates for heteroskedasticity) filter respectively.

TABLE8: Examples of risk measures using Frank copula

Equally portfolios	VaR 1%	VaR 5%	CVaR 1%	CVaR 5%
dlcac-dlsp	-0,035386	-0,0205975	-0,05315118	-0,0315992
dlcac-dldjind	-0,0364797	-0,019562	-0,04955159	-0,02949428

Although that we are far from the filtered values, the use of the more adequate dependence structure[11] gives results much closer to the empirical risk measures' values than the elliptical copula. Thus, using the "real" dependence structure can gives results that reflect the real scenarios and then it is not necessary to use the methodology of Kapil Agrawal only to have a reliable conclusion.

5. Conclusions

Our results affirm that, although the novel contribution, the study of Kapil Agrawal is restrictive. He uses only the elliptical class of copula rather than using Archimedean copula, which can take a more general dependence structure. Moreover, it is necessary to focus on data without shocks persistence, which will be done in further research.

[10] We can refer to the book of Nelson or the book of Cherubini (2004) or many articles such as those of Paul Embrechts Embrechts (1999), Embrechts (2002), Embrechts (2009)

[11] The results are much closer if the copula parameter is less than the value of 300. In the case of the portfolio of dlsp500 and dldj, we have 448.35 as a parameter value of the Franck copula; then we have a 0 as a value of the risk measures.

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