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Asian Option pricing using Path integral

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ABSTRACT

Options are traded in financial markets around the world. There are various types of options; one of them is the Asian option where the pay-off depends on the average price of the underlying asset during a certain period of time. The best known option pricing formula is the Black-Scholes' formula for European options based on the assumption of Geometric Brownian motion as a model for the asset price process. Different types of underlying assets have different characteristics, leading to different models. In order to determine option prices, advanced mathematics must be considered. Experts have found that financial markets and quantum mechanics demonstrate some common characteristics. Path integrals can represent a transitional probability density function and can be applied to any distribution of an underlying asset. Nevertheless, a solution techniques of Path integral requires tedious calculations and its closed-form does not exist yet. Monte Carlo can be applied to determining the Path integral. We investigate the weighted Monte Carlo method applied to Asian options.

1. Asian Option Pricing

Under the actual probability measure P, in an ideal market a dividend-reinvested risky asset S_t is modeled by an Ito stochastic differential equation driven by Brownian motion Z_t

$$\frac{dS_t}{S_t}=(\mu-\delta)dt+\sigma dZ_t, \quad S_{t=0}=S_0, t>0 \tag{1}$$

with μ, σ constant

In order to price derivative securities, one uses Girsanov's theorem to change P into a risk neutral (Martingale) probability measure Q"

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$$dW_t=\alpha dt+dZ_t, \ t\geq 0 \tag{2}$$

Let

$$\alpha=\frac{\mu-r}{\sigma} \tag{3}$$

We get a stochastic differential equation (SDE) with Brownian motion under risk neutral Q

$$\frac{dS_t}{S_t}=(r-\delta)dt+\sigma dW_t \tag{4}$$

where r is the risk free rate. Asset price will be the same no matter the type of investor risk preference.

$$x_t=\ln S_t$$

$$dx_t=\lambda(x_t)dt+\nu(x_t)dW_t \tag{5}$$

The pay-off of an Asian option depends on the historical average of the underlying asset price at a number of discrete monitoring points of time. The most Asian options are of the European-style. There are two subtypes of Asian option: the average price (AP) or floating price option and the average strike (AS) or floating strike option. Using *N* asset price values at equally spaced discrete time points, the payoff functions of Asian calls (C) and puts (P) at maturity time *T* are as follows:

$C_N^{AP}(S)=\max(A_N-K,0)$, $P_N^{AP}(S)=\max(K-A_N,0)$

$C_N^{AS}(S)=\max(S_N-A_N,0)$, $P_N^{AS}(S)=\max(A_N-S_N,0)$

where

$$A_N=\frac{1}{N}\sum_{n=1}^{N} S_N, \qquad S_n=S(t_0+n\frac{T-t_0}{N}) \quad (6)$$

2. Introduction to Quantum Finance

Quantum mechanics describes the world in a probabilistic manner using wave characteristic to explain particles location and velocity under a Heisenberg Uncertainty Relation. This relation states that it is impossible to measure particle's position and momentum simultaneously with unlimited accuracy. Quantum mechanics is widely used for explaining any movement using the Schrodinger equation, which has common characteristics with the Black-Scholes equation in finance. Both fields deal with randomness and uncertainties and can be characterized by stochastic process. In finance, randomness comes from many buyers and sellers that yielding many degrees of freedoms. In the quantum world, randomness comes from the measurement process. Uncertainties in quantum mechanics occur in two variables: Position and Momentum whereas in finance, they are stock price and stock velocity. Many variables in Schrodinger equation have similar characteristics to those in the Black-Scholes equation and both are differential equations. For these reasons, some experts are trying to use the quantum mechanics structure to describe financial assets behavior. One of those solutions is Feynman's Path integral, which is widely used for explaining particle movement characteristics.

3. Why Path integral

1. Every stochastic differential equation has a Feynman Path integral representation, including asset pricing model.
2. Path integral can be numerically evaluated by Monte Carlo simulation, a deterministic discretization scheme and the multinomial lattice method.
3. The Feynman Path integral can be used to model almost any dynamics, such as smooth motion, non-Markov systems, and stochastic processes driven by complex random noise. The “Handbook of Feynman Path integrals” by Grosche. And Steiner. has collected more than 1000 results of Path integrals found by physicists over the 60 years.
4. The Black-Scholes equation has many restrictive on assumptions which limit its capability for option valuation. In contrast, the Path integral formulation is more general and can be applied to various types of underlying asset.

4. Path integral Methodology

The valuation of a standard European option is given by

$$V(S_0,\wedge,T) = e^{-rT}E^Q[\wedge(S_T)|S_{t=0}=S_0] \quad (7)$$

Where $\wedge(S_T)$ is pay off, S is underlying asset price and T is expiration time of option contract

Assume a constant interest rate r (where $\hat{W}_t=W_t^Q$ is a standard Wiener process within the assumed risk-neutral measure Q)

One important assumption of Path integral valuation is that the stock price process must be a Martingale under risk neutral probability or driftless SDE i.e, $\lambda(x_t)dt = 0$

As a consequence of the Martingale property,

$$E^Q[M_{t_1}]=E^Q[M_{t_2}],\ t_1\neq t_2 \quad (8)$$

The general pricing formula is

$$V_t=e^{-r(T-t)}\int_0^{+\infty} dS_T\rho^Q(t,S|T,S_T)\,\Lambda_T\ ,\ T>t\geq 0 \quad (9)$$

Where $\rho^Q(t,S|T,S_T)$ is the transitional probability density function for which the Path integral will find an explicit solution.

5. Feynman Path integral

Let

$$\rho^Q(t,S|T,S_T)=\int_{x_t}^{X_T} Dx\ \exp(-A[x_t,X_T])$$

$$\int_{xI}^{xF} Dx\ \exp(-A_{IF}[x])=\lim_{N\to\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dx_1\ldots dx_{N-1}$$

$$\frac{\exp(-A_{01})}{\#_0(\varepsilon)}\cdots\frac{\exp(-A_{j,j+1})}{\#_j(\varepsilon)}\cdots\frac{\exp(-A_{N-1,N})}{\#_{N-1}(\varepsilon)},$$

$$\varepsilon=\frac{\lambda_F-\lambda_I}{N},\ \lambda_0=\lambda_I,\ \lambda_N=\lambda_F,\ \lambda_j=\lambda_I+j\varepsilon,\ x_j=x(\lambda_j),$$

$$A_{j,j+1}=\int_{\lambda_j}^{\lambda_{j+1}} d\lambda L,$$

$$\lim_{\varepsilon\to 0}A_{j,j+1}=\lim_{\varepsilon\to 0}\int_{\lambda_j}^{\lambda_{j+1}} d\lambda L\left(x(\lambda),\frac{dx(\lambda)}{d\lambda},\lambda\right)$$

$$=\lim_{\varepsilon\to 0}\varepsilon L\left(x_j,\frac{x_{j+1}-x_j}{\varepsilon},\lambda_I+j\varepsilon\right),\ j=0,\ldots,N-1 \quad (10)$$

Where $A[x_t,X_T]$ is called action. In modern physics, action describes how a wave function changes its phase.

$$A[x_t,X_T]=\int_t^T L(x_u,\dot{x}_u,u)du \quad (11)$$

where $L(x_u,\dot{x}_u,u)$ is Lagrangian which describes the occurrence rate of actions.

$$L(x_u,\dot{x}_u,u)=\frac{1}{2\sigma^2}(\dot{x}_u-a_u)^2$$

$$\dot{x}(\lambda)=\frac{dx(\lambda)}{d\lambda} \quad (12)$$

$a_t=r_t-\delta_t-\frac{1}{2}\sigma_t^2$, $x_t=\ln S_t$ and $dx_t=a_tdt+\sigma_tdW_t$

Those paths will be convex by normalization factors

$$\#j(\varepsilon,xj)=\sqrt{2\pi\sigma^2(e^{x_j})\varepsilon}\ \ \forall j=0,\ldots,N-1,\ \varepsilon=\frac{T-t}{N},\ N\geq 2 \quad (13)$$

Path integral-Black-Scholes equation

Direct computations of a Path integral are tricky and tedious. One can use one of several Van-Vleck formulas on the Gaussian Path integral (such as geometric Brownian motion, Ornstein-Uhlenbeck process, Cox-Ingersoll-Ross model). in order to evaluate a closed form solution:

$$\int_{x_t}^{X_T} Dx\exp\left\{\int_t^T\frac{1}{2\sigma^2}(\dot{x}\ a)^2du\right\}$$

$$=\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\exp\left(-\frac{(x_t-X_T+a(T-t))^2}{2\sigma^2(T-t)}\right),\ t<T \quad (14)$$

Put in (7) and

$$X_T|_{x_t=\ln S_t}\sim N_t^Q\left(\ln S_t+\left(r-\delta-\frac{1}{2}\sigma^2\right)(T-t),\ \sigma^2(T-t)\right) \quad (15)$$

$$\Lambda_T=\max(S_T-K,0)=\max(e^{X_T}-K,0)$$

We obtained the Black-Scholes equation

$$C(t,S_t)=S_te^{-\delta(T-t)}N(d_1)-Ke^{-\delta(T-t)}N(d_2)$$

$$d_1=\frac{\ln\frac{S_T}{K}+\left(r-\delta+\frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}}, \quad (16)$$

$$d_2=d_1-\sqrt{\sigma^2(T-t)},\ N(a)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a}\exp\left(-\frac{1}{2}u^2\right)du$$

6. Monte Carlo for Asian Option Pricing using Path integral on Ornstein-Uhlenbeck Model

A weighted Monte Carlo will be used for Asian option valuation.

We develop a discrete-time path dependent option in 3 steps as below:

1. We generate path $X=(x_1, x_2, x_3 ... x_N)$ at the set $\tau(T_N)$

2. Obtain path values of $F=(F_1, F_2, F_3 ... F_N)$ by applying appropriate invertible mapping:

$F_n = F(x_n)$, n= 1, 2, 3...N

3. Compute the random estimator $\theta(x_0, X)$ for $V(F_0, \Lambda_N, T_N)$ as a product of weight function, $W\left(x_0, X^k, \tau(T_N)\right)$, and payoff, $\Lambda_{r,T_N}(F(X^{(k)}))$.

The sample Monte Carlo estimate of $V(F_0, \Lambda_N, T_N)$ is computed as an average taken over 'M' independent realization of vector X

$$V(F_0,\Lambda,T)=E[\theta(x_0,X)\,|\,x_0=X(S_0)] \quad (17)$$

Weight function

We will use the Ornstein-Uhlenbeck model to represent stock characteristics.

(Campolieti & Marakov:2007)

Solving the SDE

$$dx_t=(\lambda_0-\lambda_1 x_t)dt+\nu_0 dW_t,\ t>0 \quad (18)$$

Let $\lambda_0=0$, $\lambda_1 > 0$

we get the transitional probability density function:

$$u(x,x_0,\tau)=\phi(y,a,b)=\frac{1}{\sqrt{2\pi b}}e^{\frac{-(y-a)^2}{2b}} \quad (19)$$

Where $y = x$, $a=x_0 e^{-\lambda_1\tau}$,

$$b=\frac{1-e^{-2\lambda_1\tau}}{\kappa},\quad \kappa=\frac{2\lambda_1}{\nu_0^2}$$

Define a pair of fundamental solutions on the entire real line

$$\varphi_\rho^-(x)=\exp\left(\frac{\kappa x^2}{4}\right)D_{-\nu}(-x\sqrt{\kappa}),\ \varphi_\rho^+(x)=\varphi_\rho^-(-x) \quad (20)$$

Some interval $I=(l,r)\subseteq R$ with the end points $-\infty\leq l<r\leq\infty$

The $\varphi_\rho^\pm(x)$ corresponding to increasing and decreasing functions of x and unique up to multiplicative constant in particular, we have

$$\lim_{x\to l^+}\frac{\varphi_s^+(x)}{\varphi_s^-(x)}=0,\ \lim_{x\to r^-}\frac{\varphi_s^+(x)}{\varphi_s^-(x)}=\infty \quad (21)$$

Let ρ represent an arbitrary non-negative constant and $D_{-\nu}(x)$ is Whittaker's parabolic cylinder function (Abramowitz and Stegun 1972)

The generating function is a linear combination of $\varphi_\rho^\pm(x)$

$$\hat{u}(x,\rho)=q_1\varphi_\rho^+(x)+q_2\varphi_\rho^-(x) \quad (22)$$

since one of q_1, q_2 is strictly positive.

The transitional probability density function is

$$u_\rho(x,x_0,t)=e^{-\rho t}\frac{\hat{u}(x,\rho)}{\hat{u}(x_0,\rho)}u(x,x_0,t) \quad (23)$$

We get the weight function as follows

$$W\left(x_0,X,\tau(T_N)\right)=e^{-\rho\tau(T_N)}\frac{\hat{u}(x,\rho)}{\hat{u}(x_0,\rho)} \quad (24)$$

The generated paths which have a greater likelihood of occurrence will have more weight than those which have less chance.

Pay off

Mapping F(x)

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$$F(x)=\frac{c_1\varphi_\rho^+(x)+c_2\varphi_\rho^-(x)}{q_1\varphi_\rho^+(x)+q_2\varphi_\rho^-(x)} \quad (25)$$

Where $c_1, c_2, q_1, q_2 \in R$

$$\frac{dF(x)}{dx}=\frac{DW[\varphi_\rho^+,\varphi_\rho^-](x)}{(q_1\varphi_\rho^+(x)+q_2\varphi_\rho^-(x))^2}=\frac{Dw_\rho s(x)}{\hat{u}^2(x,\rho)} \quad (26)$$

We get $|D|=|q_1c_2-q_2c_1|=\sigma_0/|w_\rho|\neq 0$

We can see that if q_1 equal to 0 we get interval $(F^{(l)},\infty)$ and if q_2 equal to 0 we get interval $(-\infty,F^{(r)})$. In other cases, Interval $\mathcal{J}$ bound within end point $F^{(l)}=\min(F^{(l^+)},F^{(r^-)})$ and $F^{(r)}=\max(F^{(l^+)},F^{(r^-)})$

$$\Lambda_{r,T}(F)=e^{-rT}\Lambda(e^{rT}F) \quad (27)$$

Substituting pay-off by (6)

We get a Monte Carlo's Asian option pricing on the Path integral as below:

$$V(F_0,\Lambda,T)\approx\frac{1}{M}\sum_{1}^{M}W\left(x_0,X^k,\tau(T_N)\right)\Lambda_{r,T_N}(F(X^{(k)})) \quad (28)$$

This paper discusses only Ornstein-Uhlenbeck process. There are many models that can be applied to Path integral such as geometric Brownian motion, the Cox-Ingersoll-Ross model, the Squared Bessel model, etc. based on an underlying asset's characteristics. Bandyopadhyay said "I believe it is possible (at least, theoretically) to represent the “real distribution” by a Feynman Path integral." Although I have not gotten the results yet, I believe it must be closer to the real value than that produced by the Black-Scholes method.

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Appendix

$\Lambda(x)$ is option pay-off

$W\left(x_0,X^k,\tau(T_N)\right)$ is weight function for weighted Monte Carlo

X is path contains generated x_n

$u_\rho(x,x_0,t)$ is transitional probability density function

$\hat{u}(x,\rho)$ is a generating function

$\varphi_\rho^+(x)$ is fundamental solution of increasing x

$\varphi_\rho^-(x)$ is fundamental solution of decreasing x

F(x) is future price of x

$D_{-v}(-x\sqrt{\kappa})$ is Whittaker parabolic cylinder function

$\theta(x_0,X)$ is random estimator

$A[x_t,X_T]$ is Feynman Path integral's action

$L(x_u,\dot{x}_u,u)$ is Feynman Path integral's Lagrangian

$\#j(\varepsilon,xj)$ is Feynman Path integral's normalization factor

r is risk-free rate

μ is expected return of underlying asset

σ is volatility of underlying asset

$X_T = \ln S_T$

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