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Impact of Skewness on the Performance of Optimal Portfolios

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ABSTRACT

The global minimum variance portfolio (GMVP) plays an important role in the Markowitz's mean -variance analysis . The usual assumption for studying the GMVP is the normality which is a symmetric model. In this paper, we study the influence of the skewness on the distributional properties of the estimated weights of optimal portfolios and on the corresponding inference procedures derived for the optimal portfolio weights assuming that the asset returns are normally distributed. It is shown that even a simple form of the skewness in the asset returns can dramatically influence the performance of optimal portfolios. In the empirical study, we apply our results to the real data of several stocks included into the Dow Jones index.

In this paper, we study the impact of skewness on the performance of the global minimum variance portfolio (GMVP). The GMVP plays an important role in the Markowitz's mean-variance analysis. This portfolio lies in the vertex of the efficient frontier which is a parabola in the mean-variance space (see e.g., Merton (1973), Bodnar and Schmid (2008b, 2009)). It is also a unique portfolio, whose weights are independent of the mean vector μ of the asset returns. Because of this property, the estimator of the GMVP weights does not suffer from the error in the means which is much more larger than error in the variances and covariances (see, e.g. Merton (1980), Best and Grauer (1993)). It makes the GMVP portfolio attractive for both practitioners and researchers of the financial sector (see recent studies of Jagannathan and Ma (2003), Bodnar and Schmid (2008a)).

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Let the observation matrix of the asset returns be

$$\mathbf{X} = \begin{pmatrix} x_{11} & \dots & x_{1p} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{np} \end{pmatrix} = (\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_p) = \begin{pmatrix} \mathbf{x}_1' \\ \dots \\ \mathbf{x}_n' \end{pmatrix},$$

where $\tilde{\mathbf{x}}_i$, $i = 1,\dots,p$ is the sample for the i^{th} asset in the portfolio, i.e. of the i^{th} column of $\mathbf{X}$, and $\mathbf{x}_j$ is the j^{th} observation vector of the asset return. Consequently, $\mathbf{x}_1,\dots,\mathbf{x}_n$ is a sample of the p-dimensional vectors of the asset returns.

Throughout the paper it is assumed that
$\mathbf{X} \sim CSN_{n,p;1,1}(\mathbf{1}_n \otimes \boldsymbol{\mu}', \mathbf{I}_n \otimes \boldsymbol{\Sigma}, \mathbf{D}' \otimes \mathbf{E}', 0, \nu)$
(matrix variate closed skew normal distribution), where $\boldsymbol{\Sigma}$ is a $p \times p$ positive definite matrix, $\boldsymbol{\mu}$ is a p-dimensional vector, $\mathbf{D}$ and $\mathbf{E}$ are arbitrary $n \times 1$ and $p \times 1$ vectors. The vector $\mathbf{E}$ is used to model the skewness of the vector of the

asset returns, while $\mathbf{D}$ incorporates the skewness of univariate samples of the asset returns. Consequently, we assume that the asset returns are not symmetrically distributed in time as well as the vector of the asset returns is not symmetrically distributed for a fixed time point t. If $\mathbf{D}=\mathbf{0}$ or $\mathbf{E}=\mathbf{0}$ then the matrix variate closed skew normal distribution simplifies to the matrix variate normal distribution with mean vector $\mathbf{1}_n \otimes \boldsymbol{\mu}'$ and covariance matrix $\mathbf{I}_n \otimes \boldsymbol{\Sigma}$, which we denote by $\mathcal{N}_{n,p}(\mathbf{1}_n \otimes \boldsymbol{\mu}', \mathbf{I}_n \otimes \boldsymbol{\Sigma})$.

The density function of $\mathbf{X}$ is defined using the fact that

$$vec(\mathbf{X}') \sim CSN_{np;1}(vec(\mathbf{1}_n' \otimes \boldsymbol{\mu}), \mathbf{I}_n \otimes \boldsymbol{\Sigma}, \mathbf{D}' \otimes \mathbf{E}', 0, v)$$

and it is given by

$$g_{np,1}(\mathbf{x}) = C\phi_{np}(\mathbf{x}; \mathbf{1}_n' \otimes \boldsymbol{\mu}, \mathbf{I}_n \otimes \boldsymbol{\Sigma})$$
$$\Phi_1((\mathbf{D}' \otimes \mathbf{E}')vec(\mathbf{x} - \mathbf{1}_n' \otimes \boldsymbol{\mu}); 0, v), \quad (1)$$

with

$$C^{-1} = \Phi_1(0; 0, v + (\mathbf{D}' \otimes \mathbf{E}')(\mathbf{I}_n \otimes \boldsymbol{\Sigma})(\mathbf{D} \otimes \mathbf{E})) = 1/2$$

where $\phi_l(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\Phi_l(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ denote the probability density function and the cumulative distribution function of the l-dimensional normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

The distribution considered is a special case of the more general family of the matrix variate skew normal distribution introduced in Gonzales-Farías et al. (2004). Note, that we assume that the asset returns are no longer independent or identically distributed. From Proposition 14 of Domínquez-Molina et al. (2007) with $A = \mathbf{e}_{n;j}' \otimes \mathbf{I}_p$ where $\mathbf{e}_{n;j} = (0,\ldots,0,1,0\ldots,0)'$, we get that $\mathbf{x}_j' \sim CSN_{1,p}(\mu', \boldsymbol{\Sigma}, d_j \mathbf{E}', 0, v_j)$ with $v_j = v + \mathbf{D}'\mathbf{D}\mathbf{E}'\boldsymbol{\Sigma}\mathbf{E} - d_j^2 \mathbf{E}'\boldsymbol{\Sigma}\mathbf{E}$. Hence, $\mathbf{x}_j \sim CSN_{p,1}(\mu, \boldsymbol{\Sigma}, d_j \mathbf{E}', 0, v_j)$. Under the additional assumption $\mathbf{D} = d\mathbf{1}_n$ it holds that the random vectors $\mathbf{x}_j$ are identically distributed. However, they are still dependent.

Using the results of Gupta (2004), we get the density function of $\mathbf{x}_j$ given by

$$f_{\mathbf{x}_j}(\mathbf{x}) = 2\phi_p(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})\Phi(d_j \mathbf{E}'(\mathbf{x} - \boldsymbol{\mu})/\sqrt{v_j}).$$

Moreover, it holds that

$$\mathbf{r} = E(\mathbf{x}_j) = \boldsymbol{\mu} + \frac{\sqrt{2}}{\sqrt{\pi}}(v + \mathbf{D}'\mathbf{D}\mathbf{E}'\boldsymbol{\Sigma}\mathbf{E})^{-1/2} d_j \mathbf{E}, \quad (2)$$

$$\mathbf{V} = Var(\mathbf{x}_j) = \boldsymbol{\Sigma} - \frac{2}{\pi}\frac{d_j^2}{v + \mathbf{D}'\mathbf{D}\mathbf{E}'\boldsymbol{\Sigma}\mathbf{E}}\mathbf{E}\mathbf{E}'. \quad (3)$$

If $\mathbf{E}=\mathbf{0}$ or $\mathbf{D}=\mathbf{0}$, we get $\mathbf{r} = \boldsymbol{\mu}$ and $\mathbf{V} = \boldsymbol{\Sigma}$ which are the same as in case of $\mathbf{X} \sim \mathcal{N}_{n,p}(\mathbf{1}_n \otimes \boldsymbol{\mu}, \mathbf{I}_n \otimes \boldsymbol{\Sigma})$. From (2) and (3), we conclude that if $\mathbf{D} \neq \mathbf{0}$ and $\mathbf{E} \neq \mathbf{0}$ then the weights of the mean-variance portfolio could be influenced by the skewness in the distribution of the asset returns. Otherwise, the weights do not change. Next, we determine the impact of the skewness on the weights of the global minimum variance portfolio.

Let $\mathbf{w}$ denote the weights of the portfolio. Then, the weights of the GMVP are obtained by minimizing the portfolio variance $\mathbf{w}'\mathbf{V}\mathbf{w}$ under the constraint $\mathbf{w}'\mathbf{1} = 1$ where $\mathbf{1}$ denotes a p-dimensional column vector of ones. The solution is given by

$$\mathbf{w}_{GMV} = \frac{\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}.$$

Because $\mathbf{V}$ is an unknown parameter of the asset return distribution, the vector of the GMVP weights cannot be calculated in practice. The investor estimates $\mathbf{V}$ by $\hat{\mathbf{V}}$ and then plugs $\hat{\mathbf{V}}$ in (4) instead of $\mathbf{V}$. We consider the sample estimator of the covariance matrix given by

$$\hat{\mathbf{V}} = \frac{1}{n-1}\sum_{j=1}^{n}(\mathbf{x}_j - \hat{\mathbf{r}})(\mathbf{x}_j - \hat{\mathbf{r}})' = \frac{1}{n-1}\mathbf{X}'\mathbf{M}\mathbf{X}$$

$$\text{with} \quad \hat{\mathbf{r}} = \frac{1}{n}\sum_{j=1}^{n}\mathbf{x}_j. \quad (5)$$

Here, $\mathbf{M} = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'$ is a symmetric idempotent matrix.

The weights of the global minimum variance portfolio are estimated by

$$\hat{\mathbf{w}}_{GMV} = \frac{\hat{\mathbf{V}}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\mathbf{V}}^{-1}\mathbf{1}}. \tag{6}$$

We consider a linear combination of the GMVP weights. Let $\mathbf{l}_i \in IR^p$, $i = 1,...,q$, $1 \le q \le p-1$, and $\mathbf{L}' = (\mathbf{l}_1,...,\mathbf{l}_q)$. We are interested in

$$\mathbf{w}_{L;q} = \mathbf{L}\,\mathbf{w}_{GMV} = \frac{\mathbf{L}\,\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} = \left(\frac{\mathbf{l}_1'\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}},...,\frac{\mathbf{l}_q'\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}\right)'. \tag{7}$$

Using the estimator (5), we get

$$\hat{\mathbf{w}}_{L;q} = \mathbf{L}\,\hat{\mathbf{w}}_{GMV} = \left(\frac{\mathbf{l}_1'\hat{\mathbf{V}}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\mathbf{V}}^{-1}\mathbf{1}},...,\frac{\mathbf{l}_q'\hat{\mathbf{V}}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\mathbf{V}}^{-1}\mathbf{1}}\right)'. \tag{8}$$

In order to derive the distributional properties of $\hat{\mathbf{w}}_{L;q}$ we need the following result.

Proposition 1 *Let* $\mathbf{X} \sim CSN_{n,p;1,1}(\mathbf{1}_n \otimes \boldsymbol{\mu}', \mathbf{I}_n \otimes \boldsymbol{\Sigma}, \mathbf{D}' \otimes \mathbf{E}', 0, v)$ *with* $n > p$. *Then* $(n-1)\hat{\mathbf{V}} \sim W_p(n-1, \boldsymbol{\Sigma})$ *(*p*-dimensional Wishart distribution with* $n-1$ *degrees of freedom and the covariance matrix* $\boldsymbol{\Sigma}$*).*

Let $t_q(m, \boldsymbol{\mu}, \boldsymbol{\Sigma})$ denote the q-dimensional multivariate t-distribution with m degrees of freedom, the location vector $\boldsymbol{\mu}$ and the scale matrix $\boldsymbol{\Sigma}$. Using the results of Proposition 1 and the proof of Theorem 1 of Bodnar and Schmid (2008a), we obtain the distribution of $\hat{\mathbf{w}}_{L;q}$ when the matrix of the asset returns follows a matrix variate closed skew normal distribution.

Theorem 1 *Let* $\mathbf{X} \sim CSN_{n,p;1,1}(\mathbf{1}_n \otimes \boldsymbol{\mu}', \mathbf{I}_n \otimes \boldsymbol{\Sigma}, \mathbf{D}' \otimes \mathbf{E}', 0, v)$ *with* $n > p > q \ge 1$. *Let* $H = (h_{ij})$ *be a* $p \times p-1$ *matrix with components* $h_{ii} = 1$, $h_{ki} = -1$ *for* $i = 1,..,p-1$, *and* 0 *otherwise. If* $rk(H) = q$ *then it follows that*

$$\hat{\mathbf{w}}_{L;q} \sim t_q\left(n-p+1, \tilde{\mathbf{w}}_{L;q}, \frac{1}{n-p+1}\frac{\mathbf{LRL}'}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\right),$$

where

$$\tilde{\mathbf{w}}_{L;q} = \frac{\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}$$

and $\mathbf{R} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}/\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}$ (9)

As a consequence of Theorem 1, we obtain some interesting results. First, we observe that $\hat{\mathbf{w}}_{L;q}$ is a biased estimator of $\mathbf{w}_{L;q}$ if $\mathbf{D} \ne \mathbf{0}$ and $\mathbf{E} \ne \mathbf{0}$. Second, the distribution of $\hat{\mathbf{w}}_{L;q}$ is independent of $\mathbf{D}$ and $\mathbf{E}$, although the distribution of the asset returns does depend on these parameters. Since the covariance function is a function of $\mathbf{D}$ and $\mathbf{E}$, we obtain that the true vector of the GMVP weights is given by

$$\tilde{\mathbf{w}}_{GMV} = \frac{\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} = \frac{\left(\boldsymbol{\Sigma} - \frac{2}{\pi}\frac{d_j^2}{v + \mathbf{D}'\mathbf{D}\mathbf{E}'\boldsymbol{\Sigma}\mathbf{E}}\mathbf{E}\mathbf{E}'\right)^{-1}\mathbf{1}}{\mathbf{1}'\left(\boldsymbol{\Sigma} - \frac{2}{\pi}\frac{d_j^2}{v + \mathbf{D}'\mathbf{D}\mathbf{E}'\boldsymbol{\Sigma}\mathbf{E}}\mathbf{E}\mathbf{E}'\right)^{-1}\mathbf{1}}. \tag{10}$$

and, hence, it cannot be estimated by replacing the matrix $\mathbf{V}$ by the sample covariance matrix $\hat{\mathbf{V}}$.

For providing further investigation on how large is the impact of the skewness on the performance of the global minimum variance portfolio, a test for linear combinations of the GMVP weights is applied. We consider the general linear hypothesis which is given by

$$H_0: \mathbf{L}\mathbf{w}_{GMV} = \mathbf{w}_0 \quad \textit{against} \quad H_1: \mathbf{L}\mathbf{w}_{GMV} \ne \mathbf{w}_0. \tag{11}$$

This means that the investor is interested in knowing whether the weights of the GMVP fulfill q linear restrictions or not. This is a very general testing problem and it includes many important special cases (cf. Greene (2003, pp.95-96)).

To test (11) Bodnar and Schmid (2008a) derived the following test statistic

$$T = \frac{n-k}{q}(\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1})(\hat{\mathbf{w}}_{L;q} - \mathbf{w}_0)'(\mathbf{L}\hat{\mathbf{R}}\mathbf{L}')^{-1}(\hat{\mathbf{w}}_{L;q} - \mathbf{w}_0). \quad (12)$$

Although this quantity is very similar to the F statistic for testing a linear hypothesis within the linear regression model, its distribution is different than in the case of a linear model. Consequently, the well-known results, obtained for the linear regression model, cannot be applied directly.

Let $F_{i,j}$ denote the F-distribution with degrees of freedom i and j and the density given by $f_{i,j}$. In the following we also use the hypergeometric function (cf. Abramowitz and Stegun (1984), ch. 15)

$${}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{i=0}^{\infty} \frac{\Gamma(a+i)\Gamma(b+i)}{\Gamma(c+i)} \frac{z^i}{i!}.$$

The technical computation of a hypergeometric function is a standard routine within many mathematical software packages like, e.g., in Mathematica.

Theorem 2 *Let* $\mathbf{X} \sim CSN_{n,p;1,1}(\mathbf{1}_n \otimes \boldsymbol{\mu}', \mathbf{I}_n \otimes \mathbf{\Sigma}, \mathbf{D}' \otimes \mathbf{E}', 0, v)$ *with* $n > p > q \geq 1$. *Let* $\widetilde{\mathbf{L}}' = (\mathbf{L}', \mathbf{1})$ *and* $rk(\widetilde{\mathbf{L}}) = q+1$. *The density of* T *is given by*

$$f_T(x) = f_{q,n-p}(x)\,(1+\lambda)^{-(n-p+q)/2}$$
$${}_2F_1\left(\frac{n-p+q}{2}, \frac{n-p+q}{2}, \frac{q}{2}; \frac{qx}{n-p+qx}\frac{\lambda}{1+\lambda}\right) \quad (13)$$

with

$$\lambda = \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}(\mathbf{w}_0 - \widetilde{\mathbf{w}}_{L;q})'(\mathbf{L}\mathbf{R}\mathbf{L}')^{-1}(\mathbf{w}_0 - \widetilde{\mathbf{w}}_{L;q}).$$

If $\mathbf{E} = \mathbf{0}$ or $\mathbf{D} = \mathbf{0}$ the null hypothesis is rejected as soon as $T > F_{q,n-p;1-\alpha}$, where $F_{q,n-p;1-\alpha}$ stands for the $1-\alpha$ quantile of the central F-distribution with q and $n-p$ degrees of freedom. However, if $\mathbf{E} \neq \mathbf{0}$ and $\mathbf{D} \neq \mathbf{0}$ the decision rule $T > F_{q,n-p;1-\alpha}$ might reject the null hypothesis with the probability larger than α although the vector of the target GMVP weights $\mathbf{w}_0$ is correctly specified.

In the empirical part, we show how the obtained results can be applied by cosnidering five stocks included into the Dow Jones index.

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