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Envy and Generalized Loss Aversion

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ABSTRACT

We suggest that, beside the classical endowment and disappointment effects, also envy can be modeled as Kahneman-Tversky loss aversion in a reference-dependent preference framework. In a risk-free setting especially suited to test for envy, a decomposition of reference-dependent utility into basic utility and a disappointment evaluation function is achieved, where basic utility is recovered by diagonalization. This feature sets our approach apart from disappointment theories of Sugden (JET 2003) and Bleichrodt (JMP 2009). Empirical evidences on the ultimatum game confirm this model and shows the presence of both inequality aversion on one's disadvantage (envy), and on one's advantage (Bellamare et al. Econometrica 2008).
We propose a common measure of loss aversion and envy for reference- dependent preferences, which generalizes the index of Köbberling and Wakker (JET 2005). The class of constant loss aversion/envy coincides with that of Kahneman and Tversky. We show that Arrow-Pratt absolute risk aversion increases by the relative marginal loss aversion. This inspires further tests of the loss aversion theory of envy.

1. Introduction

The simple message of this paper is to put forward the hypothesis that loss aver- sion, the fear of losing something one considers in his or her possession, jealousy, the fear of loosing it to somebody, and envy, the disutility generated by somebody possessing something one does not have but considers to ought to have, can all be described by the same type of reference-dependent preferences.

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We suggest that the only difference lies in the assignment of the allocation evaluated and the reference point. Loss aversion and jealousy take the status quo as the reference point, while for envy the envied person's allocation is taken as a reference point. Conversely, loss aversion can be interpreted as a kind of the decider's self-envy with regard to the status quo allocation of his alter ego. As envy and loss aversion are considered equivalent, the envy approach suggests a decomposition of the utility function into a self-contained basic utility and an envy evaluation function, which transforms utility differences to other allocations. This generalizes Kahneman and Tversky's [Kahneman & Tversky

1979][Tversky & Kahneman 1991] class of constant loss averse utility functions, but differs from other approaches to reference-dependent utility. A crucial difference to conventional reference-dependent approaches is that the basic utility function is obtained by diagonalization, instead of setting the reference point to zero. Diagonalization seems a natural choice for the envy interpretation, since absence of envy can be assumed in symmetric allocations.

While loss aversion and jealousy are often due to an endowment effect, envy emerges spontaneously when an allocation is announced, such as in the ultimatum game or the cake-cutting problem. We call an allocation envy-free for person i, if i does not strictly prefer any person's share. From this definition it follows that envy is evaluated taking the other person's share as a reference point, and not a mean value or an expected value. Moreover, envy is an impersonal effect without regard of the other person, since the games to measure envy effects can be set up anonymously without a human opponent. In this aspect, envy in the economic sense must be sharply distinguished from *schadenfreude*, with which it is often confused. Benevolent (empathy) and malevolent (*schadenfreude*) emotions are targeted towards a specific person, and they involve "deferential" preferences in the sense of [Pollak 2003], that is, preferences depending on other person's utility in the manner of a Samuelson-Bergson welfare function. Malevolent emotions depend on the harmed person's suffering, if the latter takes things easy, *schadenfreude* diminishes.

Loss aversion is generally modeled by reference-dependent preferences. They are represented by a three-place relation, where two alternative own allocations are compared with respect to a common reference point. In contrast, we model envious preferences by an ordering comparing two two-person allocations, which amounts to a four-place relation over alternatives. Obviously, reference-dependent preferences can be reduced to such a model by keeping the foreign allocation fixed. Moreover, an ordinary "basic" utility can be obtained by letting the other person's share be the same as the own; a procedure we call diagonalization.

Reference-dependent preference theories are abundant, see for example, [Gul 1991] and [Abdellaoui & Bleichrodt 2007], [Tversky & Kahneman 1991], [Sugden 2003], [Schmidt 2003] and [Bleichrodt 2009]. For all theories, representation theorems are stated resulting in a decomposition of utility into a reference-independent "basic" utility, which, in most theories, is considered representing the normative part of the utility function, and an evaluation function, which transforms it. All theories but [Tversky & Kahneman 1991] are formulated in a risk-dependent framework. This makes it difficult to apply those theories to game-theoretical settings and to separate loss-aversion effects from those due to probability transformation. As we shall see, in the case of non-constant loss aversion, Arrow-Pratt risk aversion of the reference-dependent utility function increases over that of the basic utility. To disentangle loss-aversion/envy effects on risk aversion from those due to probability transformations, a risk-free setting is needed.

A philosophical remark on how the envy interpretation sheds light on the debate between normative and descriptive economics should not be out of place here. In contrast to other approaches to interdependent preferences such as equity, fairness, or disappointment, there is no intrinsic moral value in considering envy. As one of the 'seven deadly sins' it spells a difference between the morally acceptable "basic" utility and its deviation towards envy. In Dante's Purgatory, the punishment for the envious is having his or her eyes sewed shut to prevent them from acquiring sinful (dis-)utility by observing others. We achieve the same effect less painfully by diagonalization of the utility function.

2. Terminology and Setting

We adopt a differentiated notion in the spirit of [Pollak 2003], who critizises Becker for applying the concept 'altruistic' to a class of preferences too wide. Preferences are **interdependent,** if one person's utility depends on other person's allocations. If one person's utility depends on another person's utility we speak of **deferent** preferences. We consider the opposite case, where other person's allocations are evaluated in the shoes of the decider. It is clear that non-deferent preferences could never represent altruistic or equality virtues except in the unlikely case of symmetric preferences. Altruistic preferences have to consider the preferences of the other person, otherwise an altruistic Moslem runs in danger of serving beef to a Hindu.

In this paper, we investigate two effects. In a two-person setting we speak of envy, if the other person's share being preferred over one's own causes a loss in utility. The reverse envy effect occurs, if one self having the favorable share causes the utility to diminish. Both cases grossly belong to the class of inequity aversion. In contrast to most approaches to envy and inequality, we consider the inequity of utilities of the deciders, and not the distance of the allocations. **Inequity loving** behavior falls into two categories: **Triumph** occurs if, having got a more favorable share, increases utility. The opposite case of increased utility if the other person got the better share could be called **mitfreude.**

In order to make our results applicable to a broad variety of settings with and without the presence of risk, we consider a convex subset X of an ordered vector space, which contains the zero element. We slightly adopt the usual axioms for an ordered vector space. Examples are convex subsets of R^L including 0, representing allocations of goods, or convex sets of real-valued random variables over a fixed probability space ordered by (first-order) stochastic dominance and containing all constant outcome variables. Another interesting example are spaces of Savage type of actions, consisting of mappings from circumstances into an outcome vector space, which are often considered closed under convex combinations. Here, the partial ordering is given by monotonicity, and the zero element in such a space can be identified analogously with some constant action mapping all circumstances to the zero element of the outcome space.

Definition 1. *An* ***ordered convex space*** *$(X, \leq)$ is a convex space X partially ordered by a reflexive, transitive, and anti-symmetric relation $\leq$, such that the following conditions hold.*

(i) $0 \in X$ and $x, y \in X$ implies that for $0 \leq \alpha \leq 1$, $\alpha \cdot x + (1-\alpha) \cdot y \in X$.

(ii) For any $x, y, z \in X$ and $0 \le \alpha \le 1$,

(2.1) $x \le y \to \alpha \cdot x + (1-\alpha) \cdot z \le \alpha \cdot y + (1-\alpha) \cdot z$.

(iii) *For any $x, y \in X$ and $\alpha \ge 0$, whenever $\alpha \cdot x, \alpha \cdot y \in X$,*

(2.2) $x \le y \to \alpha \cdot x \le \alpha \cdot y$.

The last condition always holds for $0 \le \alpha \le 1$, since $0 \in X$ implies that $\alpha \cdot x \in X$ such that (2.2) follows from (2.1) for $z = 0$. It is easy to show that these conditions are equivalent to the usual definition of an ordered vector space, which requires (2.3) for $w = z$, and (2.4).

Lemma 1. *For a convex subset X of a vector space with $0 \in X$, the following two statements hold.*

(i) *Given (2.2), (2.1) is equivalent to the following condition: For all $x, y, w, z \in X$ with $x + w, y + z \in X$,*

(2.2) $x \le y, w \le z \to x + w \le y + z$. *This condition is equivalent to the special case $w = z$.*

(ii) Given (2.3), (2.2) is equivalent to the following condition: For all $x \in X$ and $\lambda \ge 0$ with $\lambda \cdot x \in X$,

(2.4) $0 \le x \to 0 \le \lambda \cdot x$.

A reference-dependent utility function $u : X^2 \to \mathrm{R}$ maps every prospected out- come $x \in X$ and reference point $y \in X$ to the utility $u(x, y)$. Our framework is stronger than the standard models of reference-dependent preference, insofar as not only different outcomes are compared versus a fixed reference point, but pairs of outcome and a reference point. This differs from the approach of prospect theory [Tversky & Kahneman 1991], where the reference point is given by the status quo and remains unchanged during each step of the experiment. It is naturally, however, for experimental settings which involve dividing a certain goods among people. These two-person constant-sum games never have fixed reference points. Instead, each share is co-evaluated with the other person's share.

3. Envy-Separable Preferences

Most reference-dependent theories allow the inference of a basic utility function, which could be considered the normative part of utility, whose values on outcome and reference point are aggregated by the evaluation function to model the psychological impact. Consider the following scheme called satisfaction-change decomposition in [Sugden 2003] and generalized by [Bleichrodt 2009, Eq. (1)], where the reference-dependent utility to be written in terms of a basic utility function $\bar{u}$ and an evaluation function φ,

$$u(x, y) = \varphi(\bar{u}(y) - \bar{u}(x)).$$

It is, however, difficult to extract the basic utility function. Setting the reference point to zero with a scaling convention of $\bar{u}(0) = 0$ obtains $u(x, 0) = \varphi(-\bar{u}(x))$. This is ordinally equivalent to $-\bar{u}(x)$, if we assume that $u(x, y)$ is strictly monotoneous in X. However, there is no cardinal equivalence, and thus the measurement of $\bar{u}(x)$ is difficult outside a setting under risk.

For the consideration of envy effects, a natural way of obtaining the basic utility function is to set the reference point equal to the outcome. In this case, neither envy nor reverse envy effects should be present. This motivates the identification of the basic utility with the diagonal function.

Definition 2. The **diagonal function** *of a reference-dependent utility function*

$u : X^2 \rightarrow R$ *is given by*

$$v(x) = u(x, x).$$

With this definition, we can now define envy-, reverese-envy- and triumph-effects in terms of utility.

Definition 3. *Let* $u : X^2 \rightarrow R$ *be a reference-dependent utility function, and* v *its diagonal function. A pair* $(x, y) \in X^2$ *is called a case of* ***envy,*** *if* $v(y) > v(x)$ *and* $u(x, y) < v(x)$*. It is called a case of* ***reverse envy,*** *if* $v(y) < v(x)$ *and* $u(x, y) < v(x)$*. It is called a case of* ***triumph****, if* $v(y) < v(x)$ *and* $u(x, y) > v(x)$*. We call* u ***envy-neutral (reverse-envy-neutral, triumph-neutral)*** *if and only if no case of envy (reverse envy, triumph) exists.*

Many reference-dependent theories make explicit or implicit assumption on the decomposability of the utility function. In the light of our diagonalization procedure, we strengthen the standard condition as follows.

Definition 4. *A reference-dependent utility function* $u : X^2 \rightarrow R$ *is called* ***diagonally decomposable*** *if and only if there is a function* $U : X^2 \rightarrow R$ *such that for all* $x, y \in X$

$$(3.1)\quad u(x, y) = U(v(x), v(y)),$$ *where* v *is the diagonal function of* u*.*

It is easy to see that a necessary and sufficient condition to obtain $v(x)$ through diagonalization from (3.1) is $U(\zeta, \zeta) = \zeta$ on the range of v.

The following condition suggests that the excess (dis-)utility through reference-dependence is the same for equal utility differences to the reference point.

Definition 5. *A reference-dependent utility function* $u : X^2 \rightarrow R$ *is called* ***envy- separable*** *if and only if for all* $x, y, x', y' \in X$

$$v(y) - v(x) = v(y') - v(x') \Rightarrow u(x, y) - v(x) = u(x', y') - v(x'),$$

where v *is the diagonal function of* u*.*

Clearly, every envy-separable function is decomposable. It follows immediately from the definition that an envy-separable function can be written as

$$u(x, y) - v(x) = -\varphi(v(y) - v(x))$$

with some function φ, which we call the **disappointment function**. Without loss of generality we have prefixed it with a negative sign. Letting $x = y$ implies $\varphi(0) = 0$ by the diagonal condition. Rearranging yields

$$(3.2)\quad u(x, y) = v(x) - \varphi(v(y) - v(x)).$$

Another way to write this is

$$(3.3)\quad u(x,y) = v(x) - {}^{+}(v(y) - v(x))^{+} - \varphi^{-}(v(x) - v(y))^{+}$$

with $\varphi^{+}(\zeta) = \varphi(\zeta)$ and $\varphi^{-}(\zeta) = -\varphi(-\zeta)$ for $\zeta \geq 0$ being the positive and negative part of the disappointment function. They represent the inequality aversion to the person's disadvantage (envy), and to the person's advantage, respectively. We have used the notation

$$(\zeta)^{+} = \max(\zeta, 0).$$

Empirical studies confirm this model. The study of [Bellemare et al. 2008] used the Schmidt-Fehr utility function

$$(3.4) \quad u(x,y) = x - \alpha \cdot (y-x)^{+} - \beta \cdot (x-y)^{+},$$

on $X = R$ which is (3.3) for linear basic utility and evaluation functions. The model "is found to fit well the observed choices" by the authors. For a constant- sum ultimatum game with the other's share of $y = 1 - x$, the responder will reject an offer of $x<1-x$ if and only if $u(x, 1-x) \leq u(0,0) = 0$, or $x \leq \frac{\alpha}{1+2\alpha} \leq -\frac{1}{2}$.

Among a representative sample drawn from the Dutch population, the estimated parameter α of the responders (which we will later identify with the index of envy) ranges from 0.2 at the 10% percentile to 4.2 at the 90% percentile with a median of 0.92, corresponding to a rejection threshold ranging within 14%-44% with a median of 32%. It has to be noticed that the estimated value of β lies above that of α in every quantile·

4. A Measure of Envy and Loss Aversion

Several measures of or conditions for loss aversion have been proposed, most notably by [Kahneman & Tversky 1979] and [Köbberling & Wakker 2005] charac- terizing the case of constant loss-aversion. We generalize the measure to a function depending on the outcome and the reference point. In the following definition we denote by $d_x f(x, y)[z]$ and $d_y f(x, y)[z]$ the Gateaux-derivative of f for the first and second argument, respectively, in direction z.

Definition 6. *Let $u : X^2 \to R$ be a function, and $x, y \in X$ with $x = y$, and let there be an $\epsilon > \bar{0}$ with $\epsilon \cdot (y - x) \in X$. Assume that u is Gateaux-differentiable with respect to the second argument at (x, y) in direction $\epsilon \cdot (y - x)$, and that the diagonal function $v(x) = u(x, x)$ is Gateaux-differentiable at x in direction $\epsilon \cdot (y - x)$ with non-vanishing derivative. Then the* ***index of envy*** *is defined as*

$$\eta(x,y) = -\frac{d_y u(x,y)[\epsilon \cdot (y-x)]}{d_x v(y)[\epsilon \cdot (y-x)]}$$

If, moreover, there is a number $\bar{\eta}$ such that for each $z \in X$ with $v(z) > 0$,

$$\lim_{\epsilon \& 0} \eta(x, x + \epsilon \cdot z) = \bar{\eta}$$

is well-defined and identical, then $\eta(x, x)$ is defined as $\bar{\eta}$. The ***index of loss aversion*** *is defined as*

$$\lambda(x,y) = 1 + \eta(x,y).$$

If X = R, we can express the utility

function by its diagonal function and the envy measure

$$u(x,y)=v(x)-\int_x^y \eta(x,z)\cdot v'(z)\,dz$$

Definition 7. *A reference-dependent utility function $u : X^2 \to R$ with diagonal function v is said to be of **constant envy** if and only if there exists a real number $\bar{\eta}$ for all $x, y \in X$ with $v(y) > v(x)$, whenever there exists an $\epsilon > 0$ with $\epsilon\cdot(y - x) \in X$, $d_y u(x, y)[\epsilon \cdot (y - x)]$ exists and $d_x v(y)[\epsilon \cdot (y - x)]$ exists and is nonzero, and $\eta(x, y) = \bar{\eta}$. For a constant envy function, we call $\bar{\eta}$ the value of envy.*

Theorem 1. *A reference-dependent utility function $u : X^2 \to R$ is envy-separable, reverse-envy-neutral, triumph-neutral and of constant envy if and only if it can be written as*

(4.1) $u(x,y) = v(x) - \bar{\eta}\cdot(v(y)-v(x))^+$.

Moreover, if u is obeying the scaling convention $u(0, 0) = 0$, it induces a Kahneman- Tversky constant loss-aversion function with loss aversion index $\lambda = 1 + \bar{\eta}$, that is, for all $\Delta x \in X$ we have $u(\Delta x, 0) = \tilde{u}(\Delta x)$ with

(4.2) $$\tilde{u}(\Delta x)=\begin{cases} v(\Delta x), & \text{for } v(\Delta x)\geq 0,\\ \lambda\cdot v(\Delta x), & \text{for } v(\Delta x)\leq 0.\end{cases}$$

If $\lambda \geq 1$, (4.2) has a unique extension to an envy-separable, reverse-envy-neutral, triumph-neutral and constant envy reference-dependent utility function with the same value of envy as (4.1).

The uniqueness statement is important. It says that if $\tilde{u}(\Delta x)$ has been determined by a loss-aversion experiment, and it is of Kahneman-Tversky constant loss-aversion λ, then the reference dependent utility function in experimental envy settings without reverse-envy and triumph effects is uniquely given by (4.1) if and only if envy-separability holds with constant envy $\bar{\eta} = \lambda - 1$. This allows for an independent test of the hypothesis of equivalence of envy and loss aversion. Empirical estimations by [Tversky & Kahneman 1992] in the context of cumula- tive prospect theory for a basic utility function $v(x) = x^a$ have resulted in $a = 0.88$ and $\lambda = 2.25$, which amounts to a constant envy value of $\eta = 1.25$. It lies at the 60^{th} percentile of the estimated values for α (median 0.92) in the study of [Bellemare et al. 2008] for model (4.1). Considering the large spread of the parameters and the vastly different utility functions (linear versus exponential), this result leaves room for further investigations of the identity hypothesis.

The class of constant loss-aversion utility functions have the remarkable property that their Arrow-Pratt absolute risk-aversion coincides with that of their diagonal function. To investigate the general case, let $X = R$ and $u(x, y)$ be an envy-separable function differentiable for all $x = y \in R$. A straightforward calculation shows that the absolute risk aversion given reference point $y = x$ turns out as

$$\mathrm{ARA}_u(x\,|\,y) = -\frac{\frac{\partial^2 u}{\partial x^2}(x,y)}{\frac{\partial u}{\partial x}(x,y)}$$

$$= -\frac{\frac{\partial^2 v}{\partial x^2}(x)}{\frac{\partial v}{\partial x}(x)} + \frac{\partial v}{\partial x}(x)\cdot\frac{\frac{\partial^2 \varphi}{\partial x^2}(v(x)-v(y))}{1+\eta(x,y)}$$

$$= \mathrm{ARA}_v(x) + \frac{\frac{\partial \lambda}{\partial x}(x,y)}{\lambda(x,y)}$$

The last term is the relative marginal loss aversion. It can be regarded a measure of the difference between the absolute risk aversion a person reveals, and the absolute risk aversion it should have.

Appendix: Proofs

Proof of Lemma 1. (i),"⇐" : Let $x, y, z \in X$ and $0 \le \alpha \le 1$. Since X is convex and $0 \in X$, $\alpha \cdot x, (1-\alpha)\cdot x \in X$. Then (2.1) follows immediately from (2.3), since $\le$ is reflexive.

(i), "⇒": We first show (2.3) for the special case $w = z$. Let $x, y, z \in X$ with $x \le y$, $x + z$, $y + z \in X$ and set $\alpha = \frac{1}{2}$. Then $\alpha \cdot x$, $\alpha \cdot y$, $(1-\alpha)\cdot z \in X$, and, by (2.1), $\frac{1}{2}\cdot x + \frac{1}{2}\cdot z \le \frac{1}{2}\cdot y + \frac{1}{2}\cdot z$. Then by (2.4), $x + z \le y + z$. Now let $x, y, w, z \in X$ with $x + w$, $y + z \in X$, and $x \le y$, $w \le z$. Then $x + w \le y + w$, and $y + w \le y + z$, thus $x + w \le y + z$.

(ii): Since (2.4) is a special case of (2.2) for $x = 0$, we only have to show that (2.4) implies (2.2), given (2.3). Let $x, y \in X$ with $x \le y$ and $\alpha \ge 0$ with $\alpha \cdot x, \alpha \cdot y \in X$.

For $\alpha = 0$ there is nothing to show, thus assume $\alpha > 0$. We write $\alpha = n + \beta$ with an integer $n \ge 0$ and $0 \le \beta < 1$. We show inductively that $m \cdot x \le m \cdot y$ for all integer $0 \le m \le n$. For $m = 0$ or $m = 1$, there is nothing to show. Now, assume $(m-1)\cdot x \le (m-1)\cdot y$ for some $m \ge 1$. Since $m \le \alpha$, and $\alpha \cdot x \in X$, we find by convexity of X, $m\cdot x = \frac{m}{\alpha}\cdot(\alpha\cdot x) + 1 - \frac{m}{\alpha}\cdot 0 \in X$ and $m \cdot y \in X$ analogously.

Now by $x \le y$ and the induction assumption, $m \cdot x \le m \cdot y$ follows from (2.3). Thus $n \cdot x \le n \cdot y$, and, moreover, $\beta \cdot x \le \beta \cdot y$ follows with $z = 0$ from (2.1), which we have already proven to be a consequence of (2.3). Applying (2.3) again yields $\alpha \cdot x = n \cdot x + \beta \cdot x \le n \cdot y + \beta \cdot y = \alpha \cdot y$, what had to be shown. ∎

Proof of Theorem 1. Let $u : X^2 \to \mathrm{R}$ be a reference-dependent utility function Gateaux-differentiable for all $x, y \in X$ with $x = y$ in direction $\epsilon \cdot (y - x) \in X$ for some $\epsilon(x, y) > 0$, and let v be its Gateaux-differentiable diagonal function.

"⇒": Assume that u is envy-separable, reverse-envy neutral, triumph neutral and $\eta(x, y) = \bar{\eta}$ for all $x, y \in X$ with $v(y) > v(x)$. Envy-separability entails (3.2)

(4.3) $u(x, y) = v(x) - \varphi(v(y) - v(x))$.

From the definition of the envy measure there is an $\epsilon > 0$ with $\epsilon \cdot (y - x) \in X$ and

(4.4) $d_y u(x, y)[\epsilon \cdot (y - x)] = -\bar{\eta} \cdot d_x v(y)[\epsilon \cdot (y - x)]$.

From the representation (4.3) for envy-separable functions we obtain

$$d_y u(x, y)[\epsilon \cdot (y - x)] = -\varphi'(v(y) - v(x)) \cdot d_x v(y)[\epsilon \cdot (y - x)].$$

This implies that for all $x, y \in X$ with $v(y) > v(x)$, $\varphi'(v(y) - v(x)) = \bar{\eta}$, or

$$\varphi(v(y) - v(x)) = \bar{\eta} \cdot (v(y) - v(x)).$$

Since u is both reverse-envy neutral and triumph neutral, $v(y) < v(x)$ implies $u(x, y) = v(x)$, or $\varphi(v(y) - v(x)) = 0$. From $u(x, x) = v(x)$, we obtain $\varphi(0) = 0$ by (4.3), which in turn yields $\varphi(v(y) - v(x)) = 0$ whenever $v(y) = v(x)$. These three results form the representation

(4.5) $u(x, y) = v(x) - \bar{\eta} \cdot (v(y) - v(x))^+$.

"$\Leftarrow$": Function (4.5) is clearly envy-separable, reverse-envy neutral, and triumph neutral. By assumption, for each $x, y \in X$ with $x = y$, there is an $\epsilon > 0$ such that $\epsilon \cdot (y - x) \in X$. Then from (4.5) we obtain (4.4), which implies $\eta(x, y) = \bar{\eta}$ for all $x, y \in X$ with $v(y) > v(x)$.

Now assume $v(0) = u(0, 0) = 0$. For an arbitrary $\Delta x \in X$, we obtain from (4.5) $u(\Delta x, 0) = v(\Delta x) - \bar{\eta} \cdot (-v(\Delta x))^+$, which gives $u(\Delta x, 0) = \tilde{u}(\Delta x)$ with

(4.6) $\tilde{u}(\Delta x) = \begin{cases} v(\Delta x), & \text{for } v(\Delta x) \geq 0 \\ (1+\bar{\eta}) \cdot v(\Delta x), & \text{for } v(\Delta x) \leq 0 \end{cases}$

which is (4.2).

It remains to show that any function $\tilde{u}(\Delta x)$ of the form (4.6) with $\bar{\eta} \geq 0$ has a unique extension to a reference-dependent function of type (4.5) with constant envy $\bar{\eta}$. Clearly, (4.5) itself extends $\tilde{u}$. Assume that by

$$u'(x, y) = v'(x) - \bar{\eta}' \cdot (v'(y) - v'(x))^+$$

another extension of $\tilde{u}$ is given with envy value $\bar{\eta}$, such that $u(\Delta x, 0) = u'(\Delta x, 0) = \tilde{u}(\Delta x)$ for all $\Delta x \in X$. The function u' is of constant envy $\bar{\eta}'$, and by assumption, $\bar{\eta}' = \bar{\eta}$. We find

$$\tilde{u}(\Delta x) = v(\Delta x) - \bar{\eta} \cdot (-v(\Delta x))^+ = v'(\Delta x) - \bar{\eta} \cdot (-v'(\Delta x))^+.$$

If $v(\Delta x) \geq 0$ and $v'(\Delta x) < 0$, then $v(\Delta x) = (1 + \bar{\eta}) \cdot v'(\Delta x)$, which will imply $1 + \bar{\eta} \leq 0$. This is impossible, since $\bar{\eta} \geq 0$. Similarly, $v'(\Delta x) \geq 0$ and $v(\Delta x) < 0$ are impossible. It follows that $v(\Delta x) \geq 0 \Leftrightarrow v'(\Delta x) \geq 0$. From (4.6) we obtain $v(\Delta x) = v'(\Delta x)$ for $v(\Delta x) \geq 0$, and $(1 + \bar{\eta}) \cdot v(\Delta x) = (1 + \bar{\eta}) \cdot v'(\Delta x)$ for $v(\Delta x) < 0$. Altogether, $v(\Delta x) = v'(\Delta x)$ for all $\Delta x \in X$, and thus $u(x, y) = u'(x, y)$ for all $x, y \in X$. ∎

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