A Universal Framework for Financial and Actuarial Pricing of Risk: Myth or Reality?

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ABSTRACT

The aim of this work is to give a very brief overview of Wang's distortion function approach as a candidate to provide a universal framework for pricing financial risk and insurance risk. The contributions by Wang and by Hamada and Sherris are considered.

1. Introduction

Financial and insurance risks are becoming more integrated, but different approaches are used in finance and insurance to determine the fair value of risk. Wang (2002) points out that these differences include the following:

(1) Option pricing in finance is usually done via the Black-Scholes model. The Black-Scholes formula applies only to log-normal distributions of market returns. In actuarial pricing many other distributions are used.

(2) In finance, the price of an options is based by the minimal cost of setting up a hedge portfolio. The price of insurance is based on the actuarial present value of cost, an additional risk premium for correlation risk, parameter uncertainty and cost of capital.

(3) There is a difference in the types of data which is available for pricing. Option pricing is done, using risk-adjusted probabilities, on data of observed market prices of related financial assets. Actuarial pricing uses objective probabilities on available data of projected losses of which the amounts and likelihood need to be converted to a fair value price.

There is a need for a universal framework for pricing financial risk and insurance risk. This framework should, from a financial point of view, provide a model that at least fulfills the following requirements: The model has to be arbitrage free, it has to be well calibrated, it has to be realistic and transparent in its properties, and it has to allow for efficient implementation.

The aim of this work is to give a very brief overview of Wang's distortion function approach as a candidate to

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provide a universal framework for pricing financial risk and insurance risk. The contributions by Wang (1996, 2000, 2002) and by Hamada and Sherris (2003) are considered.

The paper is organized as follows. In §2 we consider the Black-Scholes model and recall the time zero price of a contingent claim and the price of a European call option. In §3 we consider risk measures and Wang's distortion function. The content of §4 includes some of the applications of Wang's approach to finance, as presented in Wang (1996, 2000, 2002) and in Hamada and Sherris (2003).

2. The Black-Scholes-Merton pricing model

In order to obtain the time zero price of a claim *C(T,K)* at time *T* with strike *K*, in the Black-Scholes model, we seek a self-financing portfolio with value exactly equal to *C(T,K)* at time *T*. In the absence of arbitrage, the value of the claim must be the same as the cost of construction of the replicating portfolio.

Let $(\Omega,\Sigma,(\Sigma_t)_{t\in[0,T]},P)$ be a filtered probability space. The market consists of two securities, a cash bond $\{B_t\}$ for which the risk-free interest rate is constant (i.e. $B_0 = 1$ and $B_t = e^{rt}$) and a risky asset, the price process of which we denote by $\{S_t\}$, which we assume to be $\{\Sigma_t\}$-measurable.

The dynamics of St is given by the stochastic differential equation

$$dS_t = S_t(\mu t + \sigma dW_t),$$

where μ and σ are constants and W_t is a *P*-Brownian motion. The wealth process $\{V_t\}$ is given by

$$V_t(\psi,\phi) = \psi_t B_t + \phi_t S_t,$$

where ϕ_t and ψ_t, respectively, denote the quantities of the assets B_t and S_t held at time *t*, and are Σ_t-predictable.

The discounted price process $\{\tilde{S}_t\}$ and discounted wealth process are defined by $\tilde{S}_t = e^{-rt}S_t$ and $\tilde{V}_t = e^{-rt}V_t$ respectively.

An application of Girsanov's Theorem yields that there exists a probability measure $\tilde{P}_t$, equivalent to *P*, under which the discounted stock price $\{\tilde{S}_t\}$ is a martingale; moreover, the Radon-Nikod´ym derivative of $\tilde{P}_t$ with respect to *P* is given by

$$m(t) = \left.\frac{d\tilde{P}}{dP}\right|_{\Sigma_t} = \exp\left\{-\left(\frac{\mu - r}{\sigma}\right)w_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right\}$$

such that $\mathbb{E}^{\tilde{P}}[X] = \mathbb{E}^{P}[mX]$;

$$\tilde{W}_t = W_t + \frac{\mu - r}{\sigma}t$$

is a standard Brownian motion under $\tilde{P}$; and

$$d\tilde{W}_t = dW_t + \frac{\mu - r}{\sigma}dt$$

Theorem 2.1. If *C(T,K)* $\in \Sigma_T$ *is a non-negative random variable for which* $\mathbb{E}^{\tilde{P}}[C(T,K)^2] < \infty$, *then the claim is replicable and the value at time t is given by*

$$V_t = \mathbb{E}^{\tilde{P}}[e^{-r(T-t)}C(T,K) \mid \Sigma_t].$$

The unique fair price at time zero is $V_0 = \mathbb{E}^{\tilde{P}}[e^{-rT}C(T,K)] = \mathbb{E}^{\tilde{P}}[\tilde{C}(T,K)]$.

Proof. For the convenience of the reader, we outline a proof; a more detailed account may be found in Etheridge (2002).

Let $M_t = \mathbb{E}^{\tilde{P}}[e^{-rT}C(T,K) \mid \Sigma_t]$. Then $\{M_t\}$ is a square-integrable $\tilde{P}$-martingale.

The natural filtration of the P-Brownian motion $\{W_t\}$ is the same as that for the $\tilde{P}$-Brownian motion $\{\tilde{W}_t\}$. By the Martingale Representation Theorem, there exists an $\sum_t$-predictable process θ_t such that $M_t = M_0 + \int_0^t \theta_s dW_s$.The choice

$\phi_t = \frac{\theta_t}{\sigma \tilde{S}_t}$ and $\psi_t = M_t - \phi_t \tilde{S}_t$, ,

yields a strategy (ψ, ϕ) which is both replicating (i.e. $\tilde{C}(T,K) = \phi_0 + \int_0^T \phi_u d\tilde{S}_u$) and self-financing (i.e. $\int_0^T |\psi_t| dt + \int_0^T |\phi_t|^2 dt < \infty$ (with probability one) and $\tilde{V}_t(\psi,\phi) = \phi_0 + \int_0^T \phi_u d\tilde{S}_u$). As $\{\tilde{V}_t\}$ is a martingale, the unique fair price at time zero is $V_0 = \phi_0$.

As a consequence of Theorem 2.1, the following may be obtained, as in Etheridge (2002):

Theorem 2.2. *The value at time t of a European option with payoff at maturity T equal to $C(T,K) = f(S_T)$ is $V_t = F(t, S_t)$, where*

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} f\left(x \exp\left((r - \frac{\sigma^2}{2})(T-t) + \sigma y \sqrt{T-t} \right) \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

Black-Scholes formula. Let $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$ denote the standard normal distribution.

For European calls and puts, with strike *K,* the function *F* in Theorem 2.2 can be calculated explicitly (see Etheridge (2002)). If $f(S_T) = (S_T - K)^+$, and letting $\theta = (T - t)$, the Black-Scholes formula follows:

$$F(t,x) = x\Phi(d_1) - K\Phi(d_2),$$

where $d_1 = \frac{\ln\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}}$ and

$d_2 = d_1 - \sigma\sqrt{\theta}$

3. Risk measures and Wang's distortion function

In any framework, the decision of how to quantify risk should reflect the reality of the basic underlying economic reality.

Risk measures.

Definition 3.1. *A function ρ that maps random variables X to real numbers $\rho(X)$ is called a risk measure provided that the following properties are satisfied:*

- *(Monotonicity) If $X \le Y$ with probability one, then $\rho(X) \le \rho(Y)$,*
- *(Translation Invariance) $\rho(X + \theta) = \rho(X) + \theta$ for all $\theta \in \mathbb{R}$.*

Many financial institutions use Value-at-Risk with confidence level α as risk measure. This notion is defined as follows:

Let *F* be the distribution of the random variable *X* and $\alpha \in \mathbb{R}$. Define

$$VaR_\alpha(X) = \min\{x \in \mathbb{R} : F(x) \ge \alpha\}$$

However, VaR_α lacks the basic property of diversification in investment science; i.e., the inequality $VaR_\alpha(X+Y) \le VaR_\alpha(X) + VaR_\alpha(Y)$ does not hold in general.

Arzner, Delbaen, Eber and Heath (1999) introduced the following more sophisticated risk measures:

Definition 3.2. *A risk measure ρ is called a coherent risk measure if the following properties are satisfied:*

- *(Positive Homogeneity) $\rho(\lambda X) = \lambda\rho(X)$ for all $\lambda > 0$,*
- *(Sub-additivity) $\rho(X + Y) = \rho(X) + \rho(Y)$.*

In order to accommodate the principle of diversification, the definition of VaR_α may be modified as follows

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 F^{-1}(t)dt.$$

This coherent risk measure is known as Tail *Value-at-Risk*.

Note that

$$VaR_\alpha(X) = F^{-1}(\alpha) = \int_0^\infty g_\alpha\left(1-F(t)\right)dt$$

where $g_\alpha(x) = 1_{(1-\alpha,1]}(x)$

and

$$TVaR_\alpha(X) = \int_0^\infty g_\alpha\left(1-F(t)\right)dt$$ where

$$g_\alpha(x) = \min\left\{1, \frac{x}{1-\alpha}\right\}.$$

The Choquet integral. The Choquet integral plays an important role in the theory of risk measures. We briefly look at some of its properties. The reader is referred to [5] for a detailed account of the Choquet integral.

Let S be any set and $\mathcal{F}$ any set of subsets of S. If $\nu : \mathcal{F} \to [0,\infty)$ is monotone (i.e., $A \subseteq B \Rightarrow v_g(A) \le v_g(B)$) then ν is called a capacity.

If ν is a capacity and $X : S \to \mathbb{R}$ is measureable (i.e. $\{X \ge x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$), then the *Choquet integral* of f with respect to ν is defined as

$$C_\nu(X) = \int_{-\infty}^0 \left(\nu(X) \ge x\right) - \nu(S))dx + \int_0^\infty \nu(X \ge x)dx.$$

Then C_ν has the following properties:

• (monotonicity) if $X \le Y$ a.s., then $C_\nu(X) \le C_\nu(Y)$

• (translation invariance) $C_\nu(X+\theta) \le C_\nu(X) + \theta$ (a consequence of the co-monotonic property of the Choquet integral);

i.e., C_ν is a risk measure. Moreover, the following also hold for C_ν

• (positive homogeneity) $C_\nu(\lambda X) = \lambda C_\nu(X)$ for all $\lambda > 0$,

• (sub-additivity) $C_\nu(X+Y) \le C_\nu(X) + C_\nu(Y)$ if and only if v is 2-alternating;

i.e. $\nu(A \cap B) + \nu(A \cup B) \le \nu(A) + \nu(B)$ for all $A, B \in \sum$.

Thus, the risk measure C_ν is coherent if and only if ν is 2-alternating.

Wang's distortion function. A non-decreasing function $g : [0,1] \to [0,1]$ for which $g(0) = 0$ and $g(1) = 1$ is called a *distortion function.*

Consider the set function $v_g = g \circ P$, where g is a distortion function. Then $v_g : \Sigma \to [0,1]$ is a capacity such that $v_g(\varnothing) = 0$ and $v_g(\Omega) = 0$.

The reader is referred to [4] for more details on this type of distortion operators.

Let $\Phi(x) = \int_o^x \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt$ denote the standard normal cumulative distribution.

Definition 3.3. For *any* $\alpha \in \mathbb{R}$, *define* Wang's distortion function g_α by

$$g_\alpha(p) = \Phi\left(\Phi^{-1}(p) + \alpha\right) \text{for all } p \in [0,1].$$

The parameter α *is called* the market price of risk.

Then $g_\alpha : [0,1] \to [0,1]$ is continuous for all $\alpha \in \mathbb{R}$; furthermore, g_α is convex for $\alpha < 0$ and concave for $\alpha > 0$.

Definition 3.4. *Let X be a random variable with decumulative distribution function* $S_X(x) = P[X > x]$ *and* $\alpha \in \mathbb{R}$.

(1) The risk-adjusted premium $H[X,\alpha]$ of *X is defined as the Choquet integral representation*

$$H[X,\alpha]=\int_{-\infty}^{0}\left(g_\alpha\left(S_X(x)\right)-1\right)dx+\int_{0}^{\infty}g_\alpha\left(S_X(x)\right)dx.$$

(2) Let $H[X,\alpha]=\int_{0}^{\infty}g_\alpha\left(S_X(x)\right)dx.$

The following result, as noted by [3], is useful in applications.

Lemma 3.5. *Let Z be a standard normal variable, h a continuous increasing function with range contained in* $[0,\infty)$. *If X = h(Z) and* $\alpha\in\mathbb{R}$, *then* $H[X,\alpha]=\mathbb{E}[h(Z+\alpha)]$.

4. Wang's approach and the Black-Scholes-Merton model

When the underlying security prices are log-normal, Wang's approach yields a simple form for the price of a contingent claim. In this section we consider some of the contributions of Wang and of Hamada and Sherris in this regard.

Pricing a security. Let S_t be the price of a security at time *t*, following a geometric Brownian motion. Then $S_t=S_0e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t}$, where W_t is a Brownian motion under *P*, and $S_T=h(Z)$ where $h(x)=S_0e^{(\mu-\frac{\sigma^2}{2})t+\sigma\sqrt{T}x}$. By Lemma 3.5, we get

$$\begin{aligned}H[S_T,-\alpha]&=\mathbb{E}[h(Z-\alpha)]\\&=\mathbb{E}[S_0e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z-\sigma\sqrt{T}\alpha}]\\&=S_0e^{(\mu-\frac{\sigma^2}{2})T-\sigma\sqrt{T}\alpha+\frac{\sigma^2}{2}t}\end{aligned}$$

The choice $\alpha=\frac{\mu-r}{\sigma}\sqrt{T}$, yields $\int_0^\infty g_{-\alpha}(S_T(x))dx=S_0e^{rT}$. Hence,

$S_0=e^{-rT}H(S_T,-\alpha)$ where $\alpha=\frac{\mu-r}{\sigma}\sqrt{T}$.

Price of a European call option. The pay-off at maturity *T* of a standard European call option with strike *K* is

$$C(T,K)=(S_T-K)^+$$

Hamada and Sherris noted that the latter may be written as *f(Z),* where *Z* is a standard normal random variable and $f(Z)=(S_0e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z}\text{-K})^+$, and that this choice yields the Black-Scholes formula. The details of their approach follow:
By Lemma 3.5,

$$\begin{aligned}h(C(T,K),\alpha)&=\mathbb{E}[f(Z)+\alpha)]\\&=\int_{-\infty}^{\infty}\left(S_0e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\alpha+\sigma\sqrt{T}z}-K\right)^+\frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}dz}.\end{aligned}$$

The values of *z* for which $S_0e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}z}\geq K$ determines the value of the integral.

These values are given by $[z_{\min},\infty)$, where

$$z_{\min}=\frac{\ln\left(\frac{K}{S_0}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T-\sigma\sqrt{T}\alpha}{\sigma\sqrt{T}}.$$

Hence,

$$h(C(T,K),-\alpha)=\int_{-\infty}^{\infty}\left(S_0e^{(\mu-\frac{\sigma^2}{2})T-\sigma\sqrt{T}\alpha+\sigma\sqrt{T}z}-K\right)^+\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}dz}$$

$$= \int_{z_{\min}}^{\infty} \left(S_0 e^{(\mu - \frac{\sigma^2}{2})T - \sigma\sqrt{T}\alpha + \sigma\sqrt{T}z} - K \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}dz}$$

$$= S_0 e^{\mu T - \sigma\sqrt{T}\alpha} \int_{z_{\min}}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2} \frac{1}{\sqrt{2\pi}} dz - K \int_{z_{\min}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}dz} dz$$

$$= S_0 e^{\mu T - \sigma\sqrt{T}\alpha} \left[1 - \Phi(z_{\min} - \sigma\sqrt{T}) \right] - K\left[1 - \Phi(z_{\min})\right]$$

$$= S_0 e^{\mu T - \sigma\sqrt{T}\alpha} \Phi(-z_{\min} + \sigma\sqrt{T}) - K\Phi(-z_{\min}).$$

The choice $\alpha = \frac{\mu - r}{\sigma}\sqrt{T}$, yields

$$e^{-rT} H(C(T,K), -\alpha) = S_0 \Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}T\right)}{\sigma\sqrt{T}} \right) - e^{-rT} K \Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}T\right)}{\sigma\sqrt{T}} \right),$$

which is the Black-Scholes price of the call option at time zero.

A representation in credit risk. Let S_t have a log-normal distribution and the dynamics of St are given by the stochastic differential equation $dS_t = S_t(\mu_t + \sigma dW_t)$, where W_t is a P-Brownian motion. The solution of the equation is

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}.$$

Under a change of measure, as in §2, the risk neutral probability measure $\tilde{P}$, equivalent to P and under which the discounted security prices form a martingale, we have that $\tilde{W}_t = W_t + \frac{\mu - r}{\sigma} t$ is a standard $\tilde{P}$-Brownian motion and $\tilde{d}W_t = dW_t + \frac{\mu - r}{\sigma} dt$. Consequently, $dS_t = S_t(rdt + \sigma d\tilde{W}_t)$ and

$$S_t = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma \tilde{W}_t}.$$

The change of measure moves the drift from μ to r.

It was noted by Hamada and Sherris in [3] that

$$P[S_t \le x] = P\left[\ln S_0 + (\mu - \frac{\sigma^2}{2})t + \sigma W_t \le \ln x \right] = \Phi\left(\frac{\ln\left(\frac{x}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \right)$$

and

$$Q[S_t \le x] = \Phi\left(\frac{\ln\left(\frac{x}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \right)$$

It follows from

$$\frac{\ln\left(\frac{x}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = \frac{\ln\left(\frac{x}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} + \frac{\mu - r}{\sigma}\sqrt{t}$$

$$= \Phi^{-1}(P[S_t \le x]) + \frac{\mu - r}{\sigma}\sqrt{t}$$

that

$$Q[S_t \le x] = \Phi\left(\Phi^{-1}(P[S_t \le x]) + \frac{\mu - r}{\sigma}\sqrt{t} \right).$$

In credit risk modeling, the latter representation is used where the default probability under the real world probability distribution is related to the default probability under the risk neutral distribution. The reader is referred to [3] for more details.

5. Conclusion

Wang's approach is easy to implement and to use for computations. There is evidence, as provided by [3, 6, 7, 8], that Wang's approach is a promising candidate to provide a universal framework for pricing financial and actuarial risk. Further research, which provides new explicit pricing formulas in financial mathematics via Wang's approach, would be desirable and seems worthwhile pursuing. It is, however, essential to verify that Wang's approach yields arbitrage free models in order to be a contender for a universal framework of pricing of risk.

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