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On the Fundamental Theorems of Asset Pricing

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ABSTRACT

Mathematics of Finance, which in general involves finding appropriate pricing models for all types of financial instruments, has its foundations in two theorems, called the fundamental theorems of asset pricing. The first fundamental theorem of asset pricing relates the concept of No Arbitrage to the existence of a risk-neutral pricing measure and the second links market completeness with the uniquness of a pricing measure. This paper gives a brief overview of both fundamental theorems of asset pricing.

1. Introduction

One of the main goals in mathematics of finance is the pricing of financial instruments in a fair and rational way. The most important property that any pricing model needs to satisfy is the principle of No Arbitrage, which roughly states that one is not allowed to make a profit without taking on any risk. The definition of No Arbitrage leads to the first fundamental theorem of asset pricing, which links the No Arbitrage principle to the existence of an equivalent martingale measure, also called a risk neutral measure. This measure is required to price derivatives fairly. The No Arbitrage condition can take on various forms and depending on which is used, different versions of the first fundamental theorem of asset pricing can be proved. The latest research, led by Frittelli (2004), involves the inclusion of a preference structure into the first fundamental theorem.

The second fundamental theorem of asset pricing relates market completeness to the uniqueness of an equivalent martingale measure. Initially the definition of completeness was linked to the No Arbitrage condition, which lead Artzner and Heath (1995) to a counterexample for this second fundamental theorem. Jarrow and Madan (1999) generalised the definition of completeness by looking at a different topology. This explained how the Artzner and Heath example fit in and allowed for a generalised version of the second fundamental theorem of asset pricing.

The aim of this paper is to bring together in one paper the different versions of the first and the second fundamental theorems of asset pricing and the notions associated with them.

The paper is divided into two sections, each dedicated to one of the two fundamental theorems of asset pricing. In the first section, we define the different No Arbitrage conditions and state the relevant versions of the first fundamental theorem of asset pricing. Then we look at including a preference structure into the above mentioned concepts. The second section

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first gives the history of the second fundamental theorem and then focuses on Battig and Jarrow's approach (1999) to this theorem of asset pricing.

2. First fundamental theorem of asset pricing

2.1. No Free Lunch. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space, where we assume that the filtration satisfies the usual conditions of right continuity[1] and completeness[2]. The filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is used to model the flow of information. T is a fixed time horizon, which can take on the value of infinity. The $\mathbb{R}^d$-valued càdlàg (i.e. right continuous with left limits) semimartingale $S = (S_t)_{t\in[0,T]}$, where $S_t = \{S_t^1, S_t^2, ..., S_t^d\}$, represents the discounted price process of d tradeable assets. Let $\mathbb{P}$ be the set of all probability measures equivalent to P and let x denote the initial endowment.

The investor has an initial endowment of x and there are no restrictions on the quantities he can buy, sell or short sell. A predictable process $H = (H_t)_{t\in[0,T]}$, where Ht $H_t = \{H_t^1, H_t^2, ..., H_t^d\}$, gives the amounts invested in each tradeable asset respectively.

Definition 2.1. An $\mathbb{R}^d$-valued S-integrable predictable process H is called *admissible* if there exists a constant c such that for all $t \in [0,T]$,

$$(H \cdot S)_t = \int_0^t H_s \cdot dS_s \geq c \quad P\text{-a.s.}$$

We denote the set of all admissible processes by

$$\mathcal{H} = \{H \in \mathfrak{P}(S) : (H \cdot S)_t\} \geq -c \text{ for all } t \in [0,T] \text{ and for some } c > 0\},$$

where $\mathfrak{P}(S)$ denotes the set of predictable and S-integrable processes.

The financial interpretation of c is a finite credit line, which the investor must respect in his trading. This lower bound on the investors losses, allows $(H \cdot S)_\infty$ to exist and bans so-called doubling strategies, where the losses are not bounded below. This restriction traces back to Harrison and Pliska (1981), and is now a standard assumption in the literature. This also ensures that any wealth process $X_T = x + \int_0^T H_s \cdot dS_s$ is bounded below.

Set

$$K = \{(H \cdot S)_T : H \in \mathcal{H}\}$$

and

$$C = (K - L_0(P)^+) \cap L_\infty(P) = \{f \in L_\infty(P) : f \leq g \ \ for \ \ some \ \ g \in K\}.$$

The set K represents the cone of all claims that are replicable at zero cost via admissible trading strategies. The set $(K - L_0^+(P))$, defined by

$$(K - L_0^+(P)) = \{f \in L_\infty(P) : f \leq g \quad P-a.s. \quad for \ \ some \ \ g \in K\},$$

is the cone of all claims in $L_0(P)$ that can be dominated by a replicable claim. In other words, it is the cone of super-replicable claims and consequently, C is the cone of bounded super-replicable claims.

A contingent claim $g \in L_\infty$ is super-replicable at price 0 if we can achieve some other contingent claim *f*, $f \geq g$, with zero net investment by pursuing some predictable trading strategy *H*. Thus it might be necessary to 'throw away money' to arrive at g (also known as 'free disposal').

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[1] $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{t>s} \mathcal{F}_s$

[2] If $A \subseteq B$ with $B \in \mathcal{F}$ such that $\mathbb{P}(B)=0$ then $A \in \mathcal{F}$

The notion of separating measures, which will be defined next was introduced by Bellini and Frittelli (2002).

Definition 2.2. A P-absolutely continuous probability measure Q is called a *separating measure* if $K \subseteq L_1(\Omega, \mathcal{F}, Q)$ and $\mathbb{E}_Q[k] \leq 0$ for all $k \in K$. It is called an *equivalent separating measure* if in addition $Q \in \mathbb{P}$. We denote the set of separating measures by $\mathcal{M}$.

The set $\mathcal{M}$ of separating measures can also be written as

$$\mathcal{M}=\{ Q<<P : \mathbb{E}_Q[w] \leq 0 \quad \text{for all} \quad w \in \mathrm{C}\}.$$

If S is bounded (resp. locally bounded), then M reduces to

$$\mathcal{M}=\{ Q<<P : S \text{ is a } Q\text{-martingale (resp. local martingale)}\}.$$

The first fundamental theorem of asset pricing essentially states that the existence of an equivalent separating measure is equivalent to a properly defined condition of No Arbitrage or No Free Lunch. The different notions of No Arbitrage depend on different closures of *C*. Let C denote the closure of *C* with respect to the norm topology of $L_\infty(P)$ and let $\bar{C}^*$ denote the weak-star closure of *C*. The set $\tilde{C}$ is the set of all limits of weak-star converging sequences of elements of *C*. Then the following defines the different notions of No Arbitrage with respect to these different closures.

Definition 2.3. A semimartingale *S* satisfies the condition of

(i) No *Arbitrage* (NA) if $C \cap L_\infty^+(P) = \{0\}$.

(ii) No *Free Lunch with Vanishing Risk* (NFLVR) if $\bar{C} \cap L_\infty^+(P) = \{0\}$.

(iii) No *Free Lunch with Bounded Risk* (NFLBR) if $\tilde{C} \cap L_\infty^+(P) = \{0\}$.

(iv) No *Free Lunch* (NFL) if $\bar{C}^* \cap L_\infty^+(P) = \{0\}$.

Note that (i) is equivalent to $K \cap L_\infty^+(P) = \{0\}$.

As $C \subseteq \bar{C} \subseteq \tilde{C} \subseteq \bar{C}^*$, we have that NFL $\Rightarrow$ NFLBR $\Rightarrow$ NFLVR $\Rightarrow$ NA.

The NA property has an obvious interpretation: the terminal payoffs cannot be positive for all admissible trading strategies as this implies that the investor will make a profit with probability 1, i.e. he makes a profit without taking on any risk. This cannot be allowed.

Harrison and Pliska (1981) were the first to prove the first fundamental theorem of asset pricing. They showed that for a finite probability space and a discrete time filtration, we have NA equivalent to the existence of an equivalent martingale measure. Dalang, Morton and Willinger (1990) extended this to infinite probability spaces.

In general, i.e. with continuous time, NA is too restrictive to imply the existence of an equivalent martingale measure for *S*. As this equivalent martingale measure is required for pricing, it was necessary to generalise the concept of NA. The idea to extend the very restrictive NA property by introducing a topology into the definition is due to Kreps (1981). He then proved the following version of the first fundamental theorem. Note, that he used a separability assumption in his proof, which is not necessary, as Yan (1980) showed. Hence, Delbaen and Schachermayer named the theorem after both these authors.

Theorem 2.4 (Kreps-Yan Theorem). *A locally bounded stochastic process S satisfies the condition of No Free Lunch if and only if there exists an equivalent local martingale measure.*

The process S admits a Free Lunch, if there exist a random variable $f \in L_\infty^+(P) \setminus \{0\}$ and a net $(f_\alpha)_{\alpha \in I} = (g_\alpha - h_\alpha)_{\alpha \in I}$ such that $g_\alpha = \int_0^T H_t^\alpha dS_t$, for some admissible trading strategy $H^\alpha, h_\alpha \geq 0$ and $(f_\alpha)_{\alpha\in I}$ converges to f in the weak-star topology of $L_\infty(P)$. Economically this implies that although f itself is not allowed to be of the form $\int_0^T H_t dS_t$, for some admissible *H* (this would be an arbitrage), it is required that *f* can be approximated by f_α in a suitable topology. In this approximation, people are allowed to 'throw money away', which is represented by the h_α to arrive at *f*.

Delbaen and Schachermayer (2006) were dissatisfied with the economic interpretation of NFL, as there is no control on the maximal loss obtained when using the trading strategy that gives the gain of g_α. They asked themselves the following:

(i) Can we find a requirement being closer to the original notion of NA and such that a version of the fundamental theorem of asset pricing still holds?

(ii) Can the weak* topology be replaced by a finer topology?

(iii) Is it possible to replace the net $(f_\alpha)_{\alpha\in I}$ by a sequence $(f_n)_{n=0}^\infty$?

(iv) Is it really necessary to allow for the 'throwing away of money'?

Added to the above drawbacks, if the semimartingale is not locally bounded, the Kreps-Yan theorem no longer holds. Hence, Delbaen (1992) introduced NFLBR, followed shortly by Delbaen and Schachermayer's (1994,1998) introduction of the notion of NFLVR. In NFLVR, the weak star topology is replaced by the topology of uniform convergence. Now, *S* allows for a Free Lunch with Vanishing Risk, if there exists a $f \in L_\infty^+(P)\setminus\{0\}$ and sequences $(f_n)_{n=0}^\infty = ((H^n \cdot S)_\infty)_{n=0}^\infty \in K$, where $(H^n)_{n=0}^\infty$ is a sequence of admissible integrands, and $(g_n)_{n=0}^\infty$ satisfying $g_n \leq f_n$, such that

$$\lim \| f - g_n \|_\infty = 0.$$

The term vanishing risk is explained by the fact that the negative parts $((f_n)^-)_{n=0}^\infty$ and $((g_n)^-)_{n=0}^\infty$ tend to zero uniformly.

As we are considering processes *S* with unbounded jumps, it is necessary to introduce a different type of martingale, called the sigma-martingale. The concept of sigma-martingales was introduced by C.S. Chou (1977) and M. Émery (1980), although the name was first coined by Delbaen and Schachermayer (1998).

Definition 2.5. A $\mathbb{R}^d$-valued semimartingale *S* is called a *sigma-martingale* if there exists a $\mathbb{R}^d$-valued martingale M and an M-integrable predictable $\mathbb{R}^+$-valued process φ such that $S = \varphi \cdot M$.

We will denote the set of sigma-martingales measures absolutely continuous with respect to *P* by $\mathbb{M}_\sigma$, i.e.

$$\mathbb{M}_\sigma = \{Q << P : S \text{ is a sigma-martingale under } Q\}.$$

Note that if *S* is bounded (resp. locally bounded), then

$$\mathbb{M}_\sigma = \{Q << P : S \text{ is a martingale (resp. local martingale) under } Q\}.$$

Note that every local martingale is a sigma-martingale.

With this new concept of NFLVR and the notion of sigma-martingales, Delbaen and Schachermayer (1994) proved another version of the first fundamental theorem of asset pricing. This version caters for the most general semimartingale market model.

Theorem 2.6. *For any semimartingale S, the following are equivalent:*

(i) *S satisfies NFLVR,*

(ii) $\mathbb{M}_\sigma(S) \cap \mathbb{P} \neq \varnothing$.

If the market is incomplete, the selection of a pricing measure becomes a nontrivial issue and one way to approach this problem is to take into consideration the preference of investors, i.e. their utility functions. Therefore, Frittelli (2004) introduced the concept of No Market Free Lunch. This allows a preference structure to be included in the concept of 'No Arbitrage'. This will be discussed in the next section.

2.2. No Market Free Lunch. In this next section we will consider the positive cone of L1 excluding zero, i.e. $L^+_\infty(P)\setminus\{0\}$.

We will interpret each element $w \in L^+_\infty(P)\setminus\{0\}$ as the time T payoff of a claim.

If we short sell this claim, then we will receive a positive amount now, but will have to pay back $-w$ at time *T*. But today we could also choose an admissible trading strategy, with zero (or negative) initial cost, that might 'hedge' the claim w. At time *t*, our payoff will then be $f - w$, where $f \in C$.

Let U be a certain set of utility functions $u = \mathbb{R} \to [-\infty, \infty]$. We assume that the preference " $\succeq$ " of the investors in the market under consideration can be represented by the expected utility, i.e.

$$f_1 \succeq f_2 \Leftrightarrow \mathbb{E}_Q[u(f_1)] \geq \mathbb{E}_Q[u(f_2)],$$

where $Q \in \mathbb{P}, u \in \mathbb{U}$ and $f_1, f_2 \in L_0(P)$.

Frittelli (2004) introduced the notion of a Market Free Lunch that depends on the preferences of the investors in the market.

Definition 2.7. (Frittelli (2004), Definition 3) There is Market Free Lunch with respect to $\mathbb{U}$ if for all $P \in \mathbb{P}$ and $u \in \mathbb{U}$, there exists $w \in L^+_\infty(P)\setminus\{0\}$ such that

$$\sup_{f\in C} \mathbb{E}_P[u(f-w)] \geq u(0). \qquad (1)$$

Hence, there is No Market Free Lunch (NMFL($\mathbb{U}$)) with respect to $\mathbb{U}$ if for all $w \in L^+_\infty(P)\setminus\{0\}$ there exist $P \in \mathbb{P}$ and $u \in \mathbb{U}$ such that

$$\sup_{f\in C} \mathbb{E}_P[u(f-w)] < u(0).$$

This definition clearly depends on the set of utility functions, which we choose. The above definition only makes economical sense if our utility function is nondecreasing on $\mathbb{R}$.

We will denote by int(A) the interior of a subset A with respect to the normtopology, i.e. the open set which is the union of all open subsets of A.

Consider the following families of utility functions

$\mathbb{U}_0 = \{u : \mathbb{R} \to [-\infty, \infty]$: u is non-decreasing on $\mathbb{R}$ },

$\mathbb{U}_1 = \{u \in \mathbb{U}_0 : u$ is left continuous at $0 \in \text{int}^*(\mathcal{D}(u))\}$ and

$\mathbb{U}_2 = \{u \in \mathbb{U}_0 : u$ is finite valued and concave on $\mathbb{R}\}$.

Note that $\mathbb{U}_2 \subseteq \mathbb{U}_1 \subseteq \mathbb{U}_0$ and $NMFL(\mathbb{U}_0) \Rightarrow NMFL(\mathbb{U}_1) \Rightarrow NMFL(\mathbb{U}_2)$

Proposition 2.8.

(i) $NMFL(\mathbb{U}_0) \Leftrightarrow NA$.

(ii) $NMFL(\mathbb{U}_1) \Leftrightarrow NFLVR$.

This proposition, proved by Frittelli (2004), shows that the difference, from an economic perspective, between NA and NFLVR is due to the differing preferences of the investors.

Under this new concept of NMFL, Frittelli (2004) proved another version of the first fundamental theorem of asset pricing.

Theorem 2.9. *For any semimartingale S, the following are equivalent:*

(i) S satisfies $NMFL(\mathbb{U}_2)$,

(ii) $\mathcal{M} \cap \mathbb{P} \neq \varnothing$.

Unfortunately, (ii) cannot be strengthened to the existence of an equivalent sigma-martingale measure.

3. Second fundamental theorem of asset pricing

3.1. History. Let $A \cap \{\Delta\}$ be an index set representing the primary traded assets. The money market account $\{\Delta\}$ is separated for easier reference and A is the possibly infinite set of risky securities.

The notion of market completeness is an important concept in mathematical finance. Before we can define completeness, some more notation and definitions are required.

Definition 3.1.

(i) The value of a portfolio at time t is given by

$$V_t = H_t^\Delta + \sum_{\alpha \in A} H_t^\alpha S_t^\alpha ,$$

(ii) A portfolio is *self-financing* if

$$dV_t = \sum_{\alpha \in A} H_t^\alpha dS_t^\alpha$$

(iii) A contingent claim X is *attainable*, if there exists a self-financing admissible strategy that replicates the claim.

(iv) The discounted contingent claim X is *integrable* if $\mathbb{E}[X] < \infty$.

(v) A trading strategy H is *stopping time simple* if there exists a sequence (h_i) such that

$$H = \sum_i h_i \Pi_{(\tau_{i=1}, \tau_i)} ,$$

where $0 \leq \tau_0 \leq \tau_1 \leq ...$ are stopping times.

There are various definitions of completeness. Harrison and Pliska (1983) define completeness as follows:

Definition 3.2. The security market is *complete* if every integrable claim is attainable.

They were the first to introduce the second fundamental theorem of asset pricing, which links market completeness with the uniqueness of an equivalent martingale measures. They showed that it holds for finite state markets with continuous trading in finitely many assets and continuous semimartingale price processes.

Harrison and Pliska asked themselves what properties a process S must have to yield a complete market. They did not attain a satisfactory general answer but they reduced the question to an equivalent problem in martingale theory and showed that the model is complete if and only if the discounted price process has the martingale representation property under some probability Q equivalent to P, i.e. each Q-martingale M can be written in the form

$$M = M_0 + \int HdS, ,$$

for some predictable process H. This in turn implies that if P is a singleton, then the model is complete.

Delbaen (1992) extended this to infinitely many securities trading in continuous time with bounded, continuous semimartingale price processes. Artzner and Heath (1995) use a topology to define approximate completeness and Ansel and Stricker (1994) use the definition of a hedge to define completeness. The reader is referred to the relevant references for more details.

In the presence of jump-discontinuities and infinitely many assets, Artzner and Heath provide a counterexample to the second fundamental theorem of asset pricing by constructing an economy in which markets are complete but there exist two distinct equivalent martingale measures. This created doubt in people's minds that the second fundamental theorem of asset pricing is true in general.

The above-mentioned literature considers the second fundamental theorem as a subsidiary to the first fundamental theorem. Market completeness is defined in terms of replicating value processes in arbitrage-free economies, where these value processes are given by discounted expectations under an equivalent martingale measure.

Jarrow and Madan (1999) considered the question of completeness in economies that may allow arbitrage opportunities. Fundamentally, the notion of completeness concerns the ability to replicate and not how these are valued or if the value is arbitrage-free. Thus the measure needed to formulate completeness is the true measure, not some presumed to exist equivalent martingale measure. Jarrow and Madan's generalisations does not change the known and proven results, when applying the fundamental theorems in conjunction. They were amongst the first to use operator theory to analyse completeness.

Jarrow et al. (1999) introduced a further generalisation proposing a setup for a securities market model and a definition of completeness, which is invariant under change to an equivalent probability measure. They proved this new notion of completeness in a discrete setting first, then Battig and Jarrow (1999) extended it to the continuous case.

They use the theory of linear operators between locally convex topological vector spaces. They consider two dual pairings $(\mathbb{X},\mathbb{Y})$ and $(\mathbb{C},\mathbb{M})$ of locally convex topological vector spaces and define a continuous linear operator *T* from $\mathbb{M}$ to $\mathbb{X}$. The transpose of this operator maps the dual of $\mathbb{X}$, i.e. $\mathbb{Y}$, to the dual of $\mathbb{M}$, i.e. $\mathbb{C}$.

In the next section, we will explain the above-mentioned dual definition of completeness in more detail.

3.2. Battig-Jarrow approach. In this section, we consider a continuous, finite trading horizon, represented by the interval [0, *T*], where *T* is fixed. Consider a filtration $(\mathcal{F}_t)_{t\in[o,T]}$ on a measurable space $(\Omega,\mathcal{F})$ and a collection *N* of events in $\mathcal{F}$, where the elements of *N* are the events that all traders agree cannot occur. The events in N are called the *null* sets.

By $L_\infty(\mathcal{F},\mathcal{N})$ we will denote the set of bounded, $\mathcal{F}$-measurable random variables, where random variables are considered equal if they differ only on the null sets. Note that we do not consider any probability measure.

Suppose that Traders are allowed to invest in the money market account and in a finite number of risky assets in A using self-financing, stopping-time simple trading strategies. The set of admissible trading strategies is given by

$$\tilde{H} = \left\{ (x,(H^\alpha)_{\alpha\in A}) : x \in \mathbb{R}, H_t^\alpha = \sum_{i=1}^{n_\alpha} h_{i-1}^\alpha \Pi_{(\tau_{i-1}^\alpha,\tau_i^\alpha)}(t) \right\},$$

where

- x is the time *0* value of the entire portfolio,
- $0 \le \tau_0^\alpha \le ... \le \tau_{n_\alpha}^\alpha \le 1$ are stopping times,
- H_t^α is the number of units of asset α held at time t for $t \in (0,T]$, and $H^\alpha = 0$ except for finitely many $\alpha \in A$,
- $h_i^\alpha \in L_\infty(\mathcal{F}_{\tau_i^\alpha},\mathcal{N})$.

The h_i^α's can be explained as follows: Over $(0,\tau_1^\alpha]$, h_0^α of asset α are held. At time τ_1^α the portfolio is rebalanced and the holdings change to h_1^α. At τ_2^α, rebalancing occurs again and you now hold h_2^α of asset α and so forth. The h_i^α's are bounded and can only be based on information known at the beginning of the interval τ_i^α.

Note that the self-financing condition requires the trader to hold

$$H_t^{\Delta} = x + \sum_{\alpha \in A} \int_0^t H_u^{\alpha} dS_u^{\alpha} - \sum_{\alpha \in A} H_t^{\alpha} S_t^{\alpha}$$

units in the money market account at time t 2 (0, T].

We will denote the payoff to a trading strategy $(x,(H^{\alpha})_{\alpha \in A})$ at time T by a function $\tilde{T}: \tilde{H} \to \mathbb{R}$ given by

$$\tilde{T}(x,(H^{\alpha})_{\alpha \in A}) = x + \sum_{\alpha \in A} \int_0^T H_u^{\alpha} dS_u^{\alpha}.$$

This represents the initial cost of the portfolio, x, plus the gains/losses on the risky asset over [0, T].

To formulate a definition for completeness, we need to define the space of attainable contingent claims C. Battig and Jarrow restrict this set to the set of bounded random variables, i.e. $C = L_{\infty}(\mathcal{F}_T, \mathcal{N})$.. Thus we can restrict $\tilde{T}$ to the set of attainable contingent claims. Let

$$\mathbb{H} = \tilde{\mathbb{H}} \cap \tilde{T}^{-1}(C).$$

Then T:$\mathbb{H} \to C$ and T is well-defined.

Theoretically, completeness implies that the trading strategies generate all the contingent claims, i.e. the image of T equals C. This way of defining completeness does not depend on any notion of arbitrage but only on the null sets N, which determine if two claims are considered identical. However, Battig and Jarrow realize that one cannot reasonably expect, that all contingent claims can be generated by outcomes of the set $\mathbb{H}$ of stopping time simple trading strategies. It is possible, though, that all elements in C can be approximated by sequences of stopping time simple trading strategies. This is what their definition of completeness is based on.

Definition 3.3. Given a measurable space (Ω,$\mathcal{F}$), a signed measure is a function $\mu : \mathcal{F} \to \mathbb{R}$ uch that $\mu(\varnothing) = 0$ and μ is sigma-additive.

The set of signed measures on $\mathcal{F}$ is a vector space. Furthermore, if μ is a signed measure, then $|\mu|$, defined by

$$|\mu|(E_i)=\sup\sum_{i=1}^{\infty}|\mu(E_i)|,$$

where the supremum is taken over all partitions $\{E_i\}$ of , is also a signed measure (see Rudin (1974), Ch 6). The set function $|\mu|$ is called the total variation of μ. It is also known that if μ is a signed measure, then

$$||\mu||=\mu(\Omega)$$

defines a norm on the vector space of signed measures (see Rudin (1974), Ch 6).

Denote by $\mathbb{M}_{sign}$ the set of signed measures on (Ω,$\mathcal{F}_T$), which assign zero mass to the elements of N. Economically, $\mathbb{M}_{sign}$ represents the set of possible contingent claim valuation measures held by traders. A trader using the valuation measure $\mu \in \mathbb{M}_{sign}$ assigns the value

$$\langle X, \mu \rangle = \int X d\mu = \mathbb{E}_{\mu}[X].$$

to an attainable contingent claim $X \in C$.

Using the set $\mathbb{M}_{sign}$ as the set of valuation measures implies that all traders agree on the events that cannot occur and it is possible that a strictly positive random variable is assigned negative personal value.

For a given trader, represented by $\mu \in \mathbb{M}_{sign}$, two contingent claims X and Y are said to be *approximately equal* if

$$\left|\int (X - Y) d\mu\right| < \epsilon \text{ for some small } \epsilon > 0,$$

i.e. if the difference in payoffs is approximately zero. This measure of closeness can be used to define a topology on *C,* which we will denote by τ^{μ}. This topology, however, is trader dependent, which is not an ideal property. To eliminate this dependence, we endow *C* with the coarsest topology that is finer than τ^{μ} for all $\mu \in \mathbb{M}_{sign}$ and denote it by τ. This topology is then trader independent. Thus, contingent claims are approximately equal if all traders, regardless of their valuation measure, agree that they are so.

Let

$$\mathcal{A}_T^x = \left\{ x + \sum_{\alpha \in A} \int_0^T H_u^\alpha dS_u^\alpha : (x, (H^\alpha)_{\alpha \in A}) \in \mathbb{H} \right\},$$

i.e. $\mathcal{A}_T^x =$ Im *T*. This is the set of contingent claims attainable by some stopping time simple trading strategy with initial cost x.

Definition 3.4. The market is *complete* if $\mathcal{A}_T^x$ = Im *T* is dense in *C* with respect to the topology τ.

Informally, the market is complete if, given an $X \in C$, there exists a trading strategy whose time *T* value all traders consider close to *X*.

The second fundamental theorem of asset pricing can now be extended accommodate the above definition of completeness. A traders personal valuation $\mu \in \mathbb{M}_{sign}$ of claims induces a valuation of trading strategies $(x, (H^\alpha)_{\alpha \in A}) \in \mathbb{H}$ given by

$$(T^* \mu)(x, (H^\alpha)_{\alpha \in A}) = \int T(x, (H^\alpha)_{\alpha \in A}) d\mu \quad (2)$$
$$= \mathbb{E}_\mu [T(x, (H^\alpha)_{\alpha \in A})].$$

In other words, a trader values a trading strategy by looking at the expected time *T* payoff generated by this strategy. However, the initial cost of the trading strategy $(x, (H^\alpha)_{\alpha \in A})$ is x. It could be possible that the trader's value of a trading strategy is not equal x. This represents a type of arbitrage opportunity that we want to exclude. Hence, we need to restrict our set of traders to those whose value in equation (2) is x. We will denote this set by $\mathbb{M}_{sign}^x$.

Theorem 3.5 (The generalised second fundamental theorem of asset pricing). *Assume there exists* $Q \in \mathbb{M}_{sign}^x$. *The security market is complete if and only if Q is unique in* $\mathbb{M}_{sign}^x$.

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